Darbhanga College of Engineering

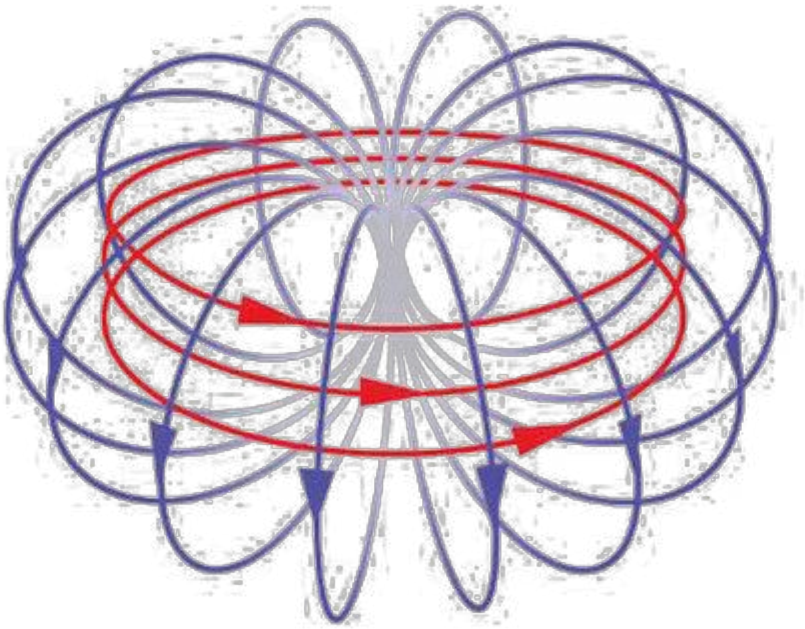
Darbhanga

Course File

Of

Electromagnetic Fields

(PCC-EEE05)



Prepared by
Dr. Ravi Ranjan
Assistant Prof.
EEE Department, DCE Darbhanga

CONTENTS

<u>Vision of the Institute</u>

To produce young, dynamic, motivated and globally competent Engineering graduates with an aptitude for leadership and research, to face the challenges of modernization and globalization, who will be instrumental in societal development.

<u>Mission of the Institute</u>

1. To impart quality technical education, according to the need of the society.
2. To help the graduates to implement their acquired Engineering knowledge for society & community development.
3. To strengthen nation building through producing dedicated, disciplined, intellectual & motivated engineering graduates.
4. To expose our graduates to industries, campus connect programs & research institutions to enhance their career opportunities.
5. To encourage critical thinking and creativity through various academic programs.

Vision of EEE Department

To bring forth engineers with an emphasis on higher studies and a fervour to serve national and multinational organisations and, the society.

Mission of EEE Department

M1: - To provide domain knowledge with advanced pedagogical tools and applications.

M2: - To acquaint graduates to the latest technology and research through collaboration with industry and research institutes.

M3: - To instil skills related to professional growth and development.

M4: - To inculcate ethical valued in graduates through various social-cultural activities.

PEO of EEE

PEO 01 – The graduate will be able to apply the Electrical and Electrical Engineering concepts to excel in higher education and research and development.

PEO 02 – The graduate will be able to demonstrate the knowledge and skills to solve real life engineering problems and design electrical systems that are technically sound, economical and socially acceptable.

PEO 03 – The graduates will be able to showcase professional skills encapsulating team spirit, societal and ethical values.

Program Educational Objectives:-
PEO 1. Graduates will excel in professional careers and/or higher education by acquiring knowledge in Mathematics, Science, Engineering principles and Computational skills.
PEO 2. Graduates will analyze real life problems, design Electrical systems appropriate to the requirement that are technically sound, economically feasible and socially acceptable.
PEO 3. Graduates will exhibit professionalism, ethical attitude, communication skills, team work in their profession, adapt to current trends by engaging in lifelong learning and participate in Research & Development.

Program Outcomes of B.Tech in Electrical and Electronics Engineering

1.Engineering knowledge: Apply the knowledge of mathematics, science, engineeringfundamentals, and an engineering specialization to the solution of complex engineering problems.

2.Problem analysis: Identify, formulate, review research literature, and analyze complexengineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.

3.Design/development of solutions: Design solutions for complex engineering problems anddesign system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.

4.Conduct investigations of complex problems: Use research-based knowledge and researchmethods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.

5.Modern tool usage: Create, select, and apply appropriate techniques, resources, and modernengineering and IT tools including prediction and modelling to complex engineering activities with an understanding of the limitations.

6.The engineer and society: Apply reasoning informed by the contextual knowledge to assesssocietal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.

7.Environment and sustainability: Understand the impact of the professional engineering solutionsin societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.

8.Ethics: Apply ethical principles and commit to professional ethics and responsibilities and norms ofthe engineering practice.

9.Individual and team work: Function effectively as an individual, and as a member or leader indiverse teams, and in multidisciplinary settings.

10.Communication: Communicate effectively on complex engineering activities with the engineeringcommunity and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.

11.Project management and finance: Demonstrate knowledge and understanding of theengineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

12.Life-long learning: Recognize the need and have the preparation and ability to engage in independent and life-long learning in the broadestcontext of technological change.

PSO 1. An ability to identify, formulate and solve problems in the areas of Electrical and Electronics Engineering.

PSO 2. An ability to use the techniques, skills and modern engineering tools necessary for innovation.

Scope and Objectives of the Course

This course is designed to understand the fundamentals of electromagnetic field theory. This course enables the students to understand all Maxwell's equation in static and time varying field. The students will also learn about Transmission line, smith Chart and reflection and refraction on plane as well oblique plane. The students will also be able to understand to solve real life problem related to electromagnetics.

Course Objectives:

The objective of this course is:

1. To provide the basic skills required to understand, develop, and design various engineering applications involving electromagnetic fields.
2. 2. To lay the foundations of electromagnetism and its practice in modern communications such as wireless, guided wave principles such as fiber optics and electronic electromagnetic structures.

Course Outcomes:

On completion of this course, the students will be able to

1. Understand electric and magnetic fields and apply the principles of Coulomb's Law and Gauss's law to electric fields in various coordinate systems.
2. Analyze Maxwell's equation in different forms (differential and integral) and apply them to diverse engineering problems.
3. Formulate and Examine the phenomena of wave propagation in different media and its interfaces and in applications of microwave engineering.
4. Analyze the nature of electromagnetic wave propagation in guided medium which are used in microwave applications.
5. Identify the electrostatic boundary-value problems by application of Poisson's and Laplace's equations.

Mapping of CO's with PO's

	PO 1	PO 2	PO 3	PO 4	PO 5	PO 6	PO 7	PO 8	PO 9	PO 10	PO 11	PO 12	PSO 1	PSO 2
CO1	3	2	2	1	1	-	2	-	-	-	1	2	3	1
CO2	2	3	3	2	3	-	-	-	-	-	1	1	2	3
CO3	2	2	3	1	3	-	-	-	-	-	1	1	1	3
CO4	2	2	1	3	3	1	1	1	-	-	2	2	2	3
CO5	1	2	1	3	-	-	3	1	2	1	-	-	2	3

Syllabus

PCC-EEE05	Electromagnetic Fields	3L:1T:0P	4 credits

Course Outcomes:

At the end of the course, students will demonstrate the ability

- To understand the basic laws of electromagnetism.
- To obtain the electric and magnetic fields for simple configurations under static conditions.
- To analyse time varying electric and magnetic fields.
- To understand Maxwell's equation in different forms and different media.
- To understand the propagation of EM waves.

This course shall have Lectures and Tutorials. Most of the students find difficult to visualize electric and magnetic fields. Instructors may demonstrate various simulation tools to visualize electric and magnetic fields in practical devices like transformers, transmission lines and machines.

Module 1: Review of Vector Calculus (6 hours)

Vector algebra-addition, subtraction, components of vectors, scalar and vector multiplications, triple products, three orthogonal coordinate systems (rectangular, cylindrical and spherical). Vector calculus-differentiation, partial differentiation, integration, vector operator del, gradient, divergence a n d curl; integral theorems of vectors. Conversion of a vector from one coordinate system to another.

Module 2: Static Electric Field (6 Hours)

Coulomb's law, Electric field intensity, Electrical field due to point charges. Line, Surface and Volume charge distributions. Gauss law and its applications. Absolute Electric potential, Potential difference, Calculation of potential differences for different configurations. Electric dipole, Electrostatic Energy and Energy density.

Module 3: Conductors, Dielectrics and Capacitance (6 Hours)

Current and current density, Ohms Law in Point form, Continuity of current, Boundary conditions of perfect dielectric materials. Permittivity of dielectric materials, Capacitance, Capacitance of a two wire line, Poisson's equation, Laplace's equation, Solution of Laplace and Poisson's equation, Application of Laplace's and Poisson's equations.

Module 4: Static Magnetic Fields (5 Hours)

Biot-Savart Law, Ampere Law, Magnetic flux and magnetic flux density, Scalar and Vector Magnetic potentials. Steady magnetic fields produced by current carrying conductors.

Module 5: Magnetic Forces, Materials and Inductance (6 Hours)

Force on a moving charge, Force on a differential current element, Force between differential current elements, Nature of magnetic materials, Magnetization and permeability, Magnetic boundary conditions, Magnetic circuits, inductances and mutual inductances.

Module 6: Time Varying Fields and Maxwell's Equations (5 Hours)

Faraday's law for Electromagnetic induction, Displacement current, Point form of Maxwell's equation, Integral form of Maxwell's equations, Motional Electromotive forces. Boundary Conditions.

Module 7: Electromagnetic Waves (6 Hours)

Derivation of Wave Equation, Uniform Plane Waves, Maxwell's equation in Phasor form, Wave equation in Phasor form, Plane waves in free space and in a homogenous material. Wave equation for a conducting medium, Plane waves in lossy dielectrics, Propagation in good conductors, Skin effect. Poynting theorem.

Module 8: Transmission line (4 Hours)

Introduction, Concept of distributed elements, Equations of voltage and current, Standing waves and impedance transformation, Lossless and low-loss transmission lines, Power transfer on a transmission line, Analysis of transmission line in terms of admittances, Transmission line calculations with the help of Smith chart, Applications of transmission line, Impedance matching using transmission lines.

Text/References:

1. M. N. O. Sadiku, "Elements of Electromagnetics", Oxford University Publication, 2014.
2. A. Pramanik, "Electromagnetism - Theory and applications", PHI Learning Pvt. Ltd, New Delhi, 2009.
3. A. Pramanik, "Electromagnetism-Problems with solution", Prentice Hall India, 2012.
4. G.W. Carter, "The electromagnetic field in its engineering aspects", Longmans, 1954.
5. W.J. Duffin, "Electricity and Magnetism", McGraw Hill Publication, 1980.
6. W.J. Duffin, "Advanced Electricity and Magnetism", McGraw Hill, 1968.
7. E.G. Cullwick, "The Fundamentals of Electromagnetism", Cambridge University Press, 1966.
8. B. D. Popovic, "Introductory Engineering Electromagnetics", Addison-Wesley Educational
   Publishers, International Edition, 1971.
9. W. Hayt, "Engineering Electromagnetics", McGraw Hill Education, 2012.

DARBHANGA COLLEGE OF ENGINEERING, DARBHANGA

Electrical and Electronics Engineering
Semester – 3th, Session (2019-23)

Monday : 09 AM – 11 AM

Thursday : 11 AM – 01 PM

DARBHANGA COLLEGE OF ENGINEERING, DARBHANGA
3th Sem. Branch:- Electrical & Electronics Engineering Batch- (2019-23)

S.No.	Name of Student	Class Roll	Registration No.
1	SHASHIBHUSHAN RAM	18EE02	18110111005
2	NITESH KUMAR PASWAN	18EE05	18110111013
3	MANJU KUMARI	18EE25	18110111020
4	AMRENDRA KUMAR	18EE47	18110111032
5	VIVEK KUMAR	18EE64	18110111039
6	DIPANSHU KUMAR	18EE77	18110111042
7	ABHISHEK KUMAR	18EE72	18110111043
8	ARCHNA KUMARI	18EE78	18110111047
9	ABHISHEK KUMAR	19EE54	19110111001
10	VIVEK KUMAR	19EE26	19110111002
11	ANKIT KUMAR	19EE08	19110111005
12	MD ASIF HUSSAIN	19EE30	19110111006
13	ABHISHEK RAJ	19EE20	19110111007
14	AMAN KUMAR	19EE34	19110111008
15	RANI KUMARI	19EE37	19110111009
16	SWATI SUMAN	19EE11	19110111010
17	AVINASH KUMAR	19EE28	19110111011
18	NITU KUMARI	19EE13	19110111012
19	RITI KUMARI	19EE18	19110111013
20	SAURAV BHUSHAN	19EE60	19110111014
21	UDIT KUMAR RANJAN	19EE32	19110111016
22	PRATYUSH KUMAR	19EE29	19110111017
23	NAYAN YADAV	19EE27	19110111018
24	ARUN KUMAR	19EE46	19110111019
25	HARSHIT RAJ	19EE01	19110111020
26	ATHARVA ADITYA	19EE47	19110111021
27	SAMI KUMAR	19EE31	19110111022
28	SAMEER KUMAR	19EE12	19110111023
29	JYOTI ANGEL	19EE52	19110111024
30	SAUMYA KUMARI	19EE53	19110111025
31	AMIT KUMAR CHAUDHARY	19EE58	19110111026
32	VIBHOOTI KUMAR	19EE09	19110111027
33	ANAND KUMAR	19EE51	19110111028
34	GOVIND KUMAR	19EE42	19110111029
35	RICHA SHUKLA	19EE38	19110111030
36	JAYHIND KUMAR	19EE40	19110111031

37	HARSH RAJ	19EE45	19110111032
38	ROUSHAN RAJ	19EE44	19110111033
39	CHANDRAKANT KUMAR	19EE61	19110111034
40	BINIT KUMAR PASWAN	19EE49	19110111035
41	ASHISH KUMAR	19EE45	19110111038
42	SHRUTI KUMARI	19EE17	19110111039
43	MANISH KUMAR	19EE05	19110111040
44	APARNA RAJ LAXMI	19EE41	19110111041
45	SHIVANI KUMARI	19EE22	19110111042
46	SAURABH KUMAR	19EE03	19110111043
47	AKSHAY KUMAR THAKUR	19EE02	19110111044
48	PREM PRAKASH	19EE21	19110111045
49	SONI KUMARI	19EE07	19110111046
50	AARTI KUMARI	19EE59	19110111047
51	CHANDAN KUMAR	19EE43	19110111048
52	SONU KUMAR	19EE15	19110111049
53	MD REHAN SHAKEEL	19EE24	19110111050
54	MD AQUBAL HUSSANI	19EE62	19110111051
55	SIDDHARTH SUMAN	19EE56	19110111052
56	APURWA KASHYAP	19EE25	19110111053
57	PRIYA RANI	19EE35	19110111054
58	DURGESH KUMAR THAKUR	19EE14	19110111055
59	AADITYA KUMAR	19EE19	19110111056
60	RISHI RANJAN	19EE10	19110111057
61	HIMANSHU KUMAR	19EE39	19110111058
62	PRIYANSHU KUMAR	19EE57	19110111059
63	ANJALI KUMARI	20LE-EE02	20110111901
64	HARSH KUMAR	20LE-EE01	20110111902
65	MANISH KUMAR PRASAD	20LE-EE14	20110111903
66	ABHISHK KUMAR	20LE-EE05	20110111904
67	HIMANSHU KUMAR	20LE-EE11	20110111905
68	ADITYA KUMAR	20LE-EE04	20110111906
69	SANTOSH KUMAR	20LE-EE10	20110111907
70	ADITYA KUMAR	20LE-EE13	20110111908
71	RAKESH KUMAR JHA	20LE-EE12	20110111909
72	ANJALI KUMARI	20LE-EE03	20110111910
73	POOJA KUMARI	20LE-EE08	20110111911
74	KAJAL KUMARI	20LE-EE07	20110111912
75	SUBHASH KUMAR	20LE-EE06	20110111913
76	SHYAM KUMAR	20LE-EE09	20110111914

Institute/College Name:	Darbhanga College of Engineering
Program Name:	B.Tech (EEE, 3^{th} semester)
Course Code:	041603
Course Name:	Electromagnetic fields
Lecture/Tutorial(per week):	4/1
Course Credits:	3
Course Co-coordinator Name:	Dr. Ravi Ranjan

Lecture Plan

Topics	No. of Lectures	Lecture Date
Module 1: Review of Vector Calculus (6 hours)		
Vector algebra-addition, subtraction, components of vectors,	**1**	
Scalar and vector multiplications, triple products,	**2**	
Three orthogonal coordinate systems (rectangular, cylindrical and spherical).	**3**	
Vector calculus-differentiation, partial differentiation, integration, vector operator del, gradient,	**4**	
Divergence and curl; integral theorems of vectors.	**5**	
Conversion of a vector from one coordinate system to another.	**6**	
Module 2: Static Electric Field (6 Hours)		
Coulomb's law, Electric field intensity,	**7**	
Electrical field due to point charges. Line, Surface and Volume charge distributions. .	**8**	
Gauss law and its applications.	**9**	
Absolute Electric potential, Potential difference	**10**	
Calculation of potential differences for different configurations.	**11**	
Electric dipole, Electrostatic Energy and Energy density	**12**	
Module 3: Conductors, Dielectrics and Capacitance (6 Hours)		
Current and current density, Ohms Law in Point form,	**13**	
Continuity of current, Boundary conditions of perfect dielectric materials.	**14**	
Permittivity of dielectric materials,	**15**	

Capacitance, Capacitance of a two wire line, Poisson's equation,	**16**	
Laplace's equation, Solution of Laplace and Poisson's equation,	**17**	
Application of Laplace's and Poisson's equations.	**18**	
Module 4: Static Magnetic Fields (5 Hours)		
Biot-Savart Law,	**19**	
Ampere Law	**20**	
Magnetic flux and magnetic flux density,	**21**	
Scalar and Vector Magnetic potentials.	**22**	
Steady magnetic fields produced by current carrying conductors.	**23**	
Module 5: Magnetic Forces, Materials and Inductance (6 Hours)		
Force on a moving charge,	**24**	
Force on a differential current element,	**25**	
Force between differential current elements,	**26**	
Nature of magnetic materials,	**27**	
Magnetization and permeability,	**28**	
Magnetic boundary conditions, Magnetic circuits, inductances and mutual inductances.	**29**	
Module 6: Time Varying Fields and Maxwell's Equations (5 Hours)		
Faraday's law for Electromagnetic induction	**30**	
Displacement current, Point form of Maxwell's equation,	**31**	
Integral form of Maxwell's equations	**32**	
Motional Electromotive forces.	**33**	
Boundary Conditions.	**34**	
Module 7: Electromagnetic Waves (6 Hours)		
Derivation of Wave Equation,	**35**	
Uniform Plane Waves, Maxwell's equation in Phasor form, Wave equation in Phasor form,	**36**	
Plane waves in free space and in a homogenous material.	**37**	
Wave equation for a conducting medium,	**38**	
Plane waves in lossy dielectrics,	**39**	
Propagation in good conductors, Skin effect. Poynting theorem.	**40**	

Module 8: Transmission line (4 Hours)		
Introduction, Concept of distributed elements, Equations of voltage and current,	**41**	
Standing waves and impedance transformation, Lossless and low-loss transmission lines, Power transfer on a transmission line,	**42**	
Analysis of transmission line in terms of admittances, Transmission line calculations with the help of Smith chart,	**43**	
Applications of transmission line, Impedance matching using transmission lines.	**44**	

DARBHANGA COLLEGE OF ENGINEERING, DARBHANGA

Subject: Electromagnetic Fields (PCC-EEE 05)

ASSIGNMENT - 1

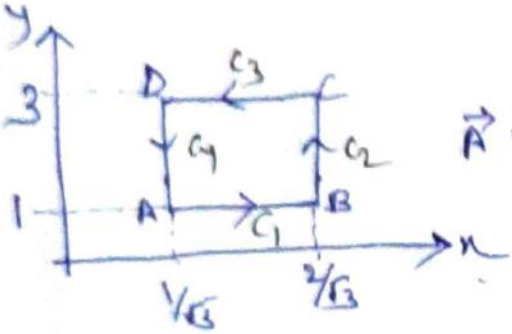
(1) Express the vector

$$B = \frac{10}{r}\hat{a}_r + r\cos\theta\,\hat{a}_\theta + \hat{a}_\phi$$

in cartesian coordinate and find $B(-3,4,0)$

Ans: $B = -2a_x + a_y$

$|B(-3,4,0)| = 2.907$

(2) 

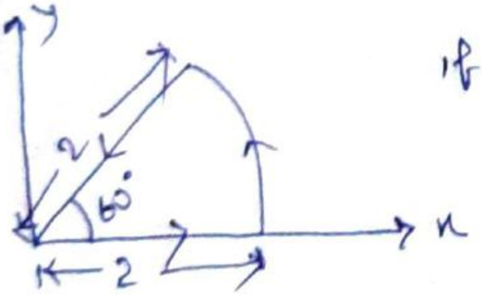
$\vec{A} = xy\,\hat{a}_x + x^2\hat{a}_y$

calculate the circulation of [redacted] $\vec{A}$ along the closed path $C(C_1 \text{ to } C_4)$

Ans: $\oint_C \vec{A}\cdot d\vec{l} = 1$

(3) Find the unit vector normal to the scalar field $y^2 = 8x$ at $(1,2)$

Ans:- $\hat{a}_{G(1,2)} = \dfrac{-8\hat{i} + 4\hat{j}}{\sqrt{32}}$

(4) 

If vector A is

$$\vec{A} = \rho\cos\phi\,\hat{a}_\rho + z\sin\phi\,\hat{a}_z$$

then evaluate $\oint A\cdot dl$ around the path shown in the figure.

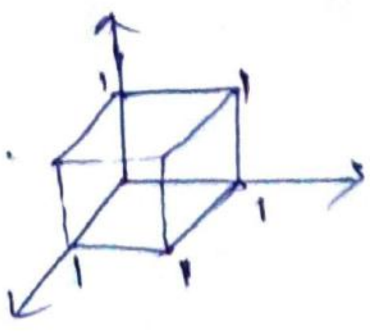
Ans: $\oint_C A\cdot dl = 1$

Hint: Can use Stoke's theorem

(5) Evaluate, $\oiint_S \vec{F}\cdot d\vec{s}$ where

$\vec{F} = 4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k}$

and S is the surface of cube formed by $0 \le x, y, z \le 1$



Ans: $\oiint_S \vec{F}\cdot d\vec{s} = 3/2$

Hint: can use Divergence theorem

DARBHANGA COLLEGE OF ENGINEERING, DARBHANGA

Subject: Electromagnetic Fields (PCC-EEE 05)

ASSIGNMENT - 2

(1) A point charge $Q_1 = 300\ \mu C$ is located at $(1, -1, -3)$ is experiencing a force of $8\hat{a}_x - 8\hat{a}_y + 4\hat{a}_z$ N due to the point charge Q_2 present at $(3, -3, -2)$. Calaculate Q_2?

Ans: $Q_2 = -40\ \mu C$

(2) A finite line charge is present along z axis $(z = \pm 5)$ with uniform density 20 nC/m. Calculate the Electric field intensity $(2, 0, 0)$.

Ans: $\vec{E} = 167.12\ V/m\ \hat{a}_x$

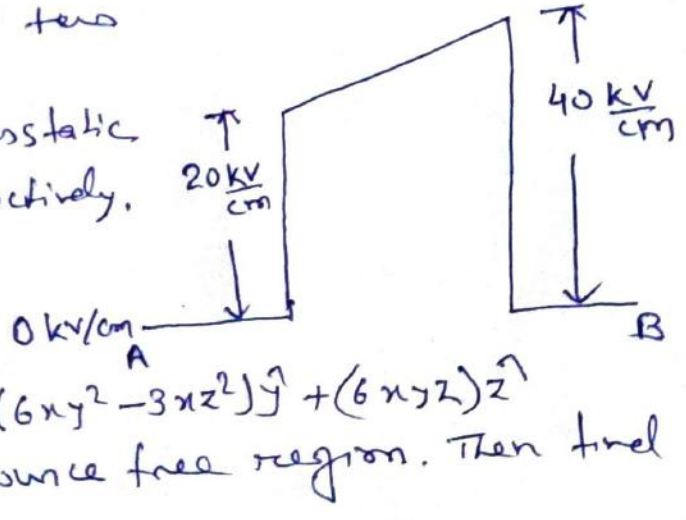
(3) Find the force experienced by 50 μC charge present at $(0, 0, 5)$ due to uniformly charged disc with 500 μC charge having radius 5 m and it is placed in $z = 0$ plane.

Ans: $\vec{F} = 16.53\ \hat{a}_z$ N

(4) potential field is given as $V = x - y + xy + 2z$
(a) Calculate $\vec{E}$ at $(1, 2, 3)$
(b) Calculate Electrostatic Energy stored in the cube of side 2 m centered at origin.

Ans: $\vec{E}_{(1,2,3)} = -3\hat{i} - 2\hat{k}$
and, $U = 16\epsilon$ Joule

(5) The Electric field (assumed to be one-dimensional) bet^n two points A and B is shown. Let V_A and V_B be the electrostatic potentials at A and B, respectively. find The value of $V_A - V_B$?



Ans: 15 Volt

(6) If $\vec{E} = -(2y^3 - 3yz^2)\hat{x} - (6xy^2 - 3xz^2)\hat{y} + (6xyz)\hat{z}$ is the Electric field in a source free region. Then find the Electrostatic potential.

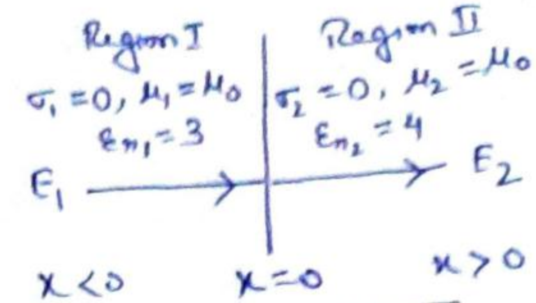
Ans: $2xy^3 - 3xyz^2$

DARBHANGA COLLEGE OF ENGINEERING, DARBHANGA

Subject: Electromagnetic Fields (PCC - EEE 05)

ASSIGNMENT - 3

Q1 A medium is divided into regions I and II about $x = 0$ plane, as shown in the figure. An Electromagnetic wave with electric field $\vec{E_1} = 4\hat{a}_x + 3\hat{a}_y + 5\hat{a}_z$ is incident normally on the interface from region-1. Find the $\vec{E_2}$ in region-II.



$\boxed{\text{Ans: } 3\hat{a}_x + 3\hat{a}_y + 5\hat{a}_z}$

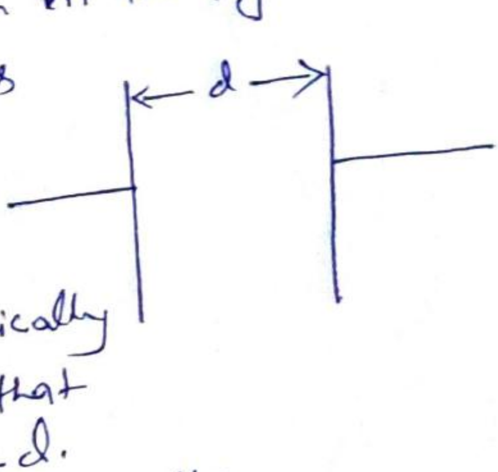
(2) Medium 1 has the electrical permittivity $\varepsilon_1 = 1.5\,\varepsilon_0$ farad/m and occupies the region to the left of $x = 0$ plane. Medium 2 has the electrical permittivity $\varepsilon_2 = 2.5\,\varepsilon_0$ farad/m and occupies the region to the right of $x = 0$ plane. If E_1 in medium 1 is $E_1 = (2u_x - 3u_y + 1u_z)$ volt/m, then find the E_2 in medium 2.

$\boxed{\text{Ans:} \rightarrow (1.2u_x - 3.0u_y + 1.0u_z)}$ ~~Ans $3\hat{a}_x + 3\hat{a}_y + 5\hat{a}_z$~~

(3) The electric field on the surface of a perfect conductor is 2 V/m. The conductor is immersed in water with $\epsilon = 80\,\epsilon_0$. Find the surface charge density on the conductor.

Ans: $\rightarrow 1.41 \times 10^{-9}$ C/m²

(4) The parallel-plate capacitor shown in the figure has movable plates. The capacitor is charged so that the energy stored in it is E when the plate separation is d. The capacitor is then isolated electrically and the plates are moved such that the plate separation becomes 2d. At this new plate separation, what is the energy stored in the capacitor.



Ans: 2E

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Subject: Electromagnetic Fields (PCC-EEE 05)

ASSIGNMENT-4

(1) A magnetic field in air is measured to be

$$\vec{B} = B_0\left(\frac{x}{x^2+y^2}\hat{y} - \frac{y}{x^2+y^2}\hat{x}\right)$$

Ans: $\vec{J} = 0$

What current distribution leads to this field?

(2) An infinitely long uniform solid wire of radius a carries a uniform dc current of density $\vec{J}$.

(a) r for $r<a$ and $1/r^2$ for $r>a$

(b) 0 for $r<a$ and $1/r$ for $r>a$

(c) r for $r<a$ and $1/r$ for $r>a$

(d) 0 for $r<a$ and $1/r^2$ for $r>a$

(3) The unit of $\nabla \times \vec{H}$ is

(a) Ampere

(b) Ampere/meter

(c) Ampere/meter^2

(d) Ampere-meter

(4) Deduce an expression for magnetic field intensity H due to an infinitely long current-carrying conductor carrying current I. Use Biot-Savart law.

(5) The magnetic field at any point on the axis of a current carrying circular coil will be:

(a) perpendicular to the axis

(b) parallel to the axis

(c) at an angle 45° with axis

(d) zero.

Code : 103307

2013 (A)

ELECTROMAGNETIC FIELD THEORY

Time : 3 hours *Full Marks* : 70

Instructions :

(i) *The marks are indicated in the right-hand margin.*

(ii) *There are **TEN** questions in this paper.*

(iii) *Attempt any **FIVE** questions.*

1. *(a)* Find the potential distribution due to a long pair of parallel wires of negligible cross-section and having equal and opposite line charge density. Also obtain equipotential surfaces produced by them.

   (b) Find the capacitance of two parallel cylindrical conductors having their radii as a and separation between their axes as b. 9+5=14

2. *(a)* State uniqueness theorem and prove it.

   (b) Explain conductor properties and obtain boundary conditions.

   (c) For a two-dimensional system in which $r = \sqrt{x^2 + y^2}$, determine $\nabla^2 V$ when $V = \frac{1}{r}$. 6+5+3=14

3. *(a)* Find the energy density in the magnetic field.

   (b) Find the magnetic field inside a solid conductor carrying a direct current, and hence obtain total magnetic flux per unit length within the conductor.

   (c) Prove Stokes' theorem. 5+5+4=14

4. *(a)* Obtain two Maxwell's equations which deviate from steady-state field.

   (b) The electric field of electromagnetic wave is given by $E_x = 0 = E_z$, $E_y = A\cos\omega\left(t - \frac{z}{c}\right)$. Using Maxwell's equation in free space, find the magnetic vector $\vec{H}$. 9+5=14

5. (a) Find the ratio of $\vec{E}$ and $\vec{H}$ in a uniform plane wave.

   (b) Discuss the wave propagation in conducting medium and obtain the value of α and β. 8+6=14

6. Derive the reflection coefficient of perfect dielectric for oblique incidence in the case of parallel polarization. Obtain Brewster angle. 14

7. (a) State Poynting theorem and prove it.

   (b) A short vertical transmitting antenna erected on the surface of a perfectly conducting earth produces effective field strength

$$E_{eff} = E_{\theta\, eff} = 100 \sin\theta \frac{mu}{m}$$

   at points at a distance of one mile from the antenna. Compute the Poynting vector and total power radiated. 9+5=14

8. (a) Discuss UHF line as circuit element and hence find the input impedance of short-circuited quarter-wave line.

   (b) Discuss quarter-wave line as transformer. 8+6=14

9. (a) Discuss Smith chart and its uses.

   (b) Design a necessary matching unit to join without impedance mismatch the two different sections of transmission line whose impedances are 75 ohm and 50 ohm. 10+4=14

10. Find the field component of TM wave in parallel plane guide and hence discuss TEM wave. 14

* * *

Code : 031506

B.Tech 5th Semester Exam., 2013

ELECTROMAGNETIC FIELD THEORY

Time : 3 hours *Full Marks* : 70

Instructions :

(i) The marks are indicated in the right-hand margin.

*(ii) There are **NINE** questions in this paper.*

*(iii) Attempt **FIVE** questions in all.*

(iv) Question No. 1 is compulsory.

1. Fill in the blanks (any *seven*) : 2×7=14

$r(\nabla \times A)$

(a) Divergence of a curl of a vector is ——.

(b) Energy density in the electrostatic field is $\frac{1}{2}\varepsilon_0 t^2$

(c) The value of relative permeability is slightly less than one for —— and slightly greater than one for ——.

(d) Tangential component of electric field is —— across the interface between two dielectric media. $E_1 \tan\theta_1 \leq \varepsilon$

(e) Surface impedance of good conductor is just equal to ~~377~~

(f) For uniform plane wave E field and H field has $\perp$ in the direction of propagation.

(g) VSWR varies from —— to ——.

(h) Short circuited quarter wave section and open end half-wave section is analogous to ——.

(i) If the standing wave of voltage slope is up towards the termination, then the reactance will be ——.

(j) The quality factor of a resonant section of transmission line is equal to the ratio of —— per unit length to —— per unit length.

2. (a) For a two-dimensional system $r = \sqrt{x^2 + y^2}$, determine $\nabla^2 V$, when $V = \ln \frac{1}{r}$.

(b) Find out the divergence of vector and interpret it by giving physical examples.

(c) State and prove divergence theorem. 4+8+2=14

3. (a) State and prove uniqueness theorem.

(b) Find the capacitance of two spheres, whose separation d is very much larger than their radii R. Hence show that the capacitance of sphere above an infinite ground plane is independent of the height h above the plane when $h >> R$. 4+(5+5)

4. (a) Describe magnetic vector potential.

(b) Explain Ampere force law.

(c) Find the magnetic field inside a solid conductor carrying a direct current I and hence obtain total magnetic flux per unit length within the conductor. 5+3+6

5. (a) Obtain continuity equation for time-varying field.

(b) Explain in consistency of Ampere circuital law.

(c) The electric vector $\vec{E}$ of a electromagnetic wave in free space is given by the expression

$$E_y = A\cos\omega\left(t - \frac{z}{c}\right)$$

Using Maxwell's equation for free space condition, determine magnetic vector $\vec{H}$. 5+5+4

6. (a) Find the component of $\vec{E}$ and $\vec{H}$ in the direction of the propagation for uniform plane wave.

(b) Establish the relation between $\vec{E}$ and $\vec{H}$ in a uniform plane wave.

(c) Show that the function

$$F = e^{-\alpha z}\sin\frac{\omega}{v}(x - vt)$$

satisfies the wave equation

$$\nabla^2 F = \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}$$

provided that the wave velocity is given by

$$v = c\left(1 + \frac{\alpha^2 c^2}{\omega^2}\right)^{-\frac{1}{2}}$$ 4+6+4

7. (a) Find the reflection coefficient by perfect dielectric for parallel polarization and hence obtain Brewster angle.

(b) Discuss surface impedance. 11+3

8. (a) State and prove Poynting theorem.

(b) Discuss Smith chart. (4+6)+4

9. (a) Find the quality factor of a resonant transmission line section.

(b) Find the voltage step up in quarter wave line. 9+5

★ ★ ★

CONDUCTORS, DIELECTRICS AND CAPACITANCE

⇒ ELECTRIC FIELDS IN MATERIAL SPACE

So far we considered electrostatic fields in free space or a space that has no materials in it.

Now, we study the electric phenomena in material space.

Materials
- ↳ Conductors
- ↳ Non-conductors (insulators or dielectric)

↳ Properties of Materials

Conductivity σ → mhos per meter ($\mho/m$) or Siemens per meter (S/m)

↓ depends on temperature and frequency

↳ high conductivity ($\sigma >> 1$) → metal

low conductivity ($\sigma << 1$) → Insulators.

↳ The material whose conductivity lies somewhere betn metals and insulators is called a semiconductors.

↳ The conductivity of metals generally increases with decrease in temperature. At temperature near absolute zero ($T = 0K$), some conductors exhibit infinite conductivity and are called superconductors. Lead and aluminum are typical example of such metals. like, the conductivity of lead at 4K is of order of 10^{20} S/m.

↳ Convection and conduction Currents

The current (in amperes) through a given area is the electric charge passing through the area per unit time.

$$I = \frac{dQ}{dt} \qquad \text{—— (1)}$$

the current density, J (current ΔI flows through a planar surface, ΔS)

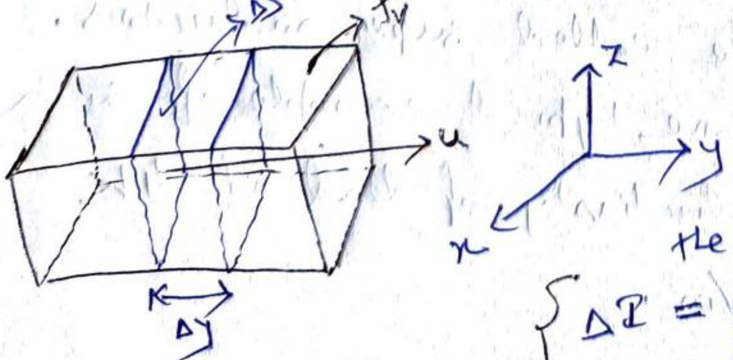
$$J = \frac{\Delta I}{\Delta S} \Rightarrow \Delta I = J \cdot \Delta S \qquad \text{—— (2)}$$

$$I = \int_S J \cdot ds \qquad \text{—— (3)}$$

↳ Depending on how I is produced, there are different kind of current density.

- → Convection current density
- → Conduction current density
- → Displacement current density

⇒ Convection current does not involve conductors and consequently does not satisfy Ohm's law. It occurs when current flows through an insulating medium such as liquid, rarefied gas or a vacuum.



If we consider a filament where the flow of charge of density ρ_v at velocity $u = u_y a_y$ (u is y dirn)

the current through the filament is:—

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \, \Delta S \, \frac{\Delta y}{\Delta t} = \rho_v \, \Delta S \, u_y \qquad \text{—— (4)}$$

The y-directed current density J_y is given by

$$J_y = \frac{\Delta I}{\Delta S} = \rho_v u_y \quad \text{——— (5)}$$

Hence, in general, $J = \rho_v u$ ——— (6)

convection current density (A/m^2)

Conduction current

It requires a conductor. A conductor is characterized by a large number of free electrons that provide conduction current due to an impressed electric field. (E)

So, $F = -eE$ (since, electron has charge $-e$) ——— (7)

If an electron with mass m is moving in an electric field $\mathbf{E}$ with an average drift velocity $\mathbf{u}$, according to Newton's law, the average change in momentum of the free electron must match the applied force

So, $F = \dfrac{mu}{\tau}$ where, τ = average time interval betn collision

So, $\dfrac{mu}{\tau} = -eE$

or, $$u = -\frac{e\tau}{m} E \quad \text{——— (8)}$$

~~of the electron charge~~ if there are n electrons per unit volume, the electron charge density given by

$$\rho_v = -ne \quad \text{——— (9)}$$

Thus the conduction current density

$$J = \rho_v u = \frac{ne^2\tau}{m} E = \sigma E \quad \text{——— (10)}$$

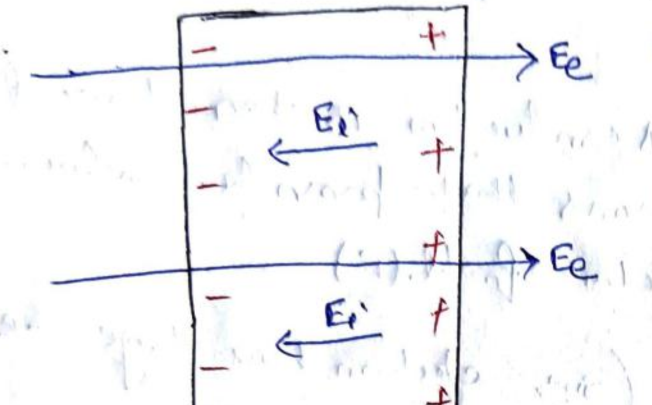
where, $\sigma = \dfrac{ne^2\tau}{m}$ is the conductivity of the conductor

So, $\boxed{J = \sigma E}$ —— (11)

it is known as point form of ohm's law

⇒ CONDUCTORS

A conductor has an abundance of charge that is free to move.



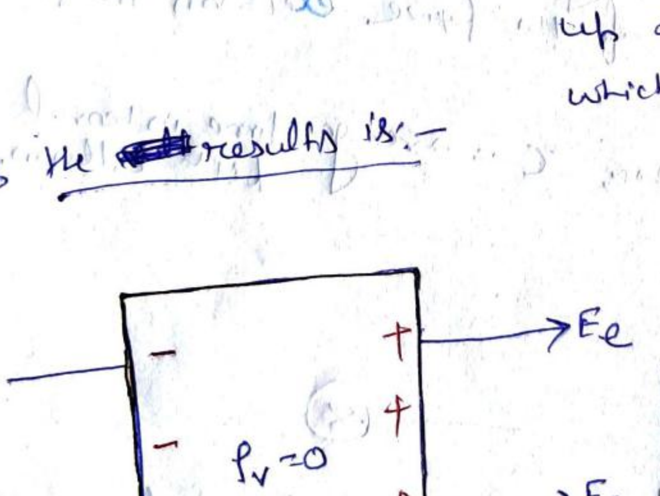
- here in isolated conductor, an external electric field E_e is applied.
  - ↳ then +ve free charges pushed along E_e (same dirn)
  - ↳ -ve free charges move opposite to E_e

so, the free charges accumulate on the surface of the conductor and form an induced ~~charge~~ surface charge.

these induced charges set up an internal induced field E_i, which cancel the externally applied field E_e.

so the ~~[illegible]~~ results is:-

{A perfect conductor ($\sigma = \infty$) cannot contain an electrostatic field within



↳ Since E inside the conductor is zero ($E_e - E_i = 0$)

$E = -\nabla V = 0$

So, V is const so, inside the conductor equipotential.

↳ Again, from ohm's law, $J = \sigma E$

to maintain, finite current density J, in a perfect conductor ($\sigma \to \infty$) we have to make $E \to 0$ so that σE = finite.

(5)

If we introduce some charge in the interior of such conductor, the charges will move to the surface and redistribut themselves quickly in such a manner that the field inside the conductor vanishes.

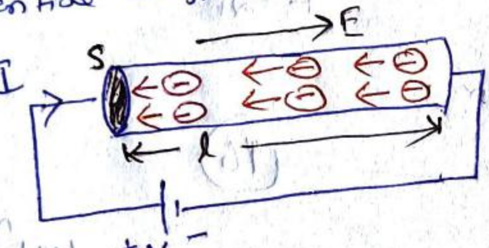
from Gauss's law

$$\rho_v = \nabla \cdot D = \nabla \cdot \epsilon E$$

So for, $E = 0$, $\rho_v = 0$, So Under the static conditions

$$\boxed{E = 0, \quad \rho_v = 0, \quad V_{ab} = 0 \text{ inside a conductor}} \quad (12)$$

i) Now, we consider a conductor whose ends are maintained at a potential difference V as shown in figure.



Here $E \neq 0$ inside the conductor and there is no static equilibrium since the conductor is not isolated but is wired to a source of electromotive force, which compels the free charge to move and prevents the electrostatic equilibrium.

So, an electric field must inside the conductor to sustain the flow of current.

As the electrons move, they encounter some damping force called resistance.

So, Here, the Electric feel applied is uniform and its magnitude is given by, $E = \frac{V}{l}$ (13)

Since, the conductor has a uniform cross section,

So, $J = \frac{I}{S}$ (14)

By substitute eqn (11) & (13) into (14)

$$\frac{I}{S} = J = \sigma E = \sigma \frac{V}{l}$$

Hence,

$$\left\{ R = \frac{V}{I} = \frac{l}{\sigma S} = \frac{\rho_c l}{S} \right\} \quad \text{—— (15)}$$

where, $\rho_c = 1/\sigma$ is the resistivity of the material

Now, power P (in watts) is defined as the rate of change of Energy W (in Joules) or force times velocity

$$P = \int E \cdot \rho_v u \, dv = \int \underbrace{E \cdot \frac{\rho_v dv}{Q}}_{F = QE} \cdot \overset{\text{drift velocity}}{u}$$

$$= \int E \cdot \underbrace{\rho_v u}_{J} \, du$$

$$\left\{ P = \int E \cdot J \, dv \right\} \quad \text{—— (16)}$$

the power density, $\left\{ w_p = \frac{dP}{dP} = E \cdot J = \sigma |E|^2 \right\} \quad \text{—— (17)}$

For a conductor with uniform cross section

$dv = ds\,dl$, so, $P = \int_L E\,dl \int_S J\,dS = VI$

or, $$\left\{ P = I^2 R = \frac{V^2}{R} \right\} \quad \text{—— (18)}$$

Here In R (resistance), we take uniform cross section, but if the cross section is not uniform, then,

$$\boxed{R = \frac{V}{I} = \frac{\int E \cdot dl}{\int_S \sigma E \cdot dS}} \quad \text{—— (19)}$$

Polarization in Dielectrics

The main difference betn a conductor and a dielectric lies in availability of free electrons in the atomic shells to conduct current.

The charges in the dielectric are not able to move about freely, they are bound by finite forces and we may certainly expect a displacement when a external force is applied.

$E = 0$

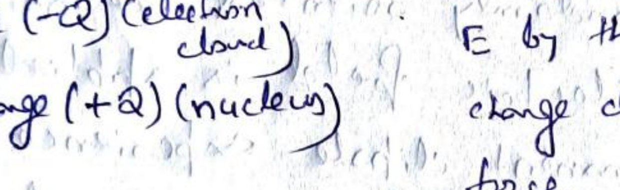
(a) A dielectric atom, −ve charge $(-Q)$ (electron cloud) and +ve charge $(+Q)$ (nucleus)

+ve charge displaced from its equilibrium position in the dirn of E by the force $F_+ = QE$ and −ve charge displaced opposite by the force, $F_- = -QE$

So, A dipole is resultant from the displacement of the charges, and the dielectric is said to be polarized.

This distorted charge distribution is equivalent, by the principal of superposition, to the original distribution plus a dipole whose moment is

$$\boxed{P = Qd} \qquad (20)$$



in the presence of E

→ So, if in the volume Δv there are N dipole created. Then the total Dipole moment due to electric field is

$$Q_1 d_1 + Q_2 d_2 + \cdots + Q_n d_N = \sum_{k=1}^{N} Q_k d_k$$

So, to measure the intensity of polarization, polarization 'P' (C/m²) as the dipole moment per unit volume of the dielectric

$$P = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{N} Q_k d_k}{\Delta v} \quad \text{——— (21)}$$

So, the major effect of the dielectric field E on a dielectric is the creation of dipole moment that align themselves in the direction of E. This type of dielectric are said to be non-polar. (like, hydrogen, oxygen, nitrogen, rare gases)

non-polar dielectric do not possess dipoles until the application of the electric field.

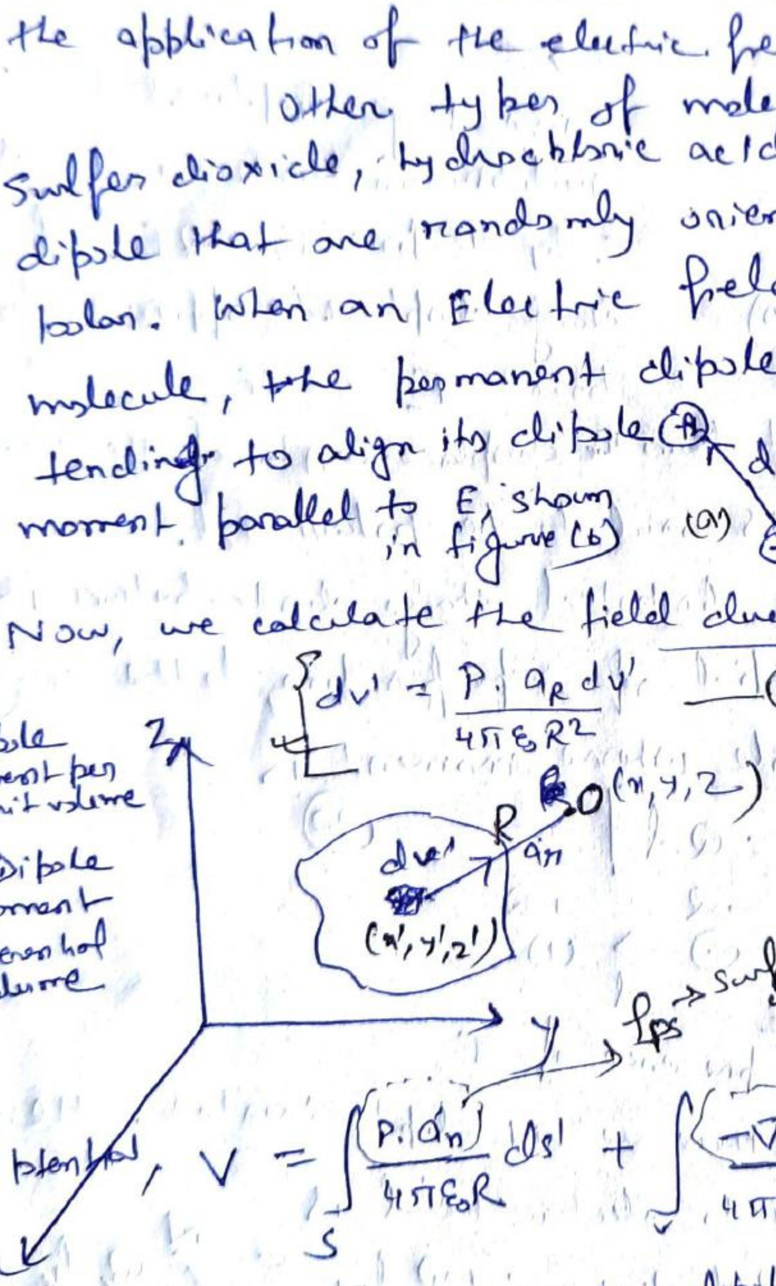
Other types of molecule such as water, sulfer dioxide, hydrochloric acid have built in permanent dipole that are randomly oriented, are said to be polar. When an Electric field is applied to polar molecule, the permanent dipole experience a torque tending to align its dipole moment parallel to E, shown in figure (b)

→ Now, we calculate the field due to polarized dielectric.

$$dV' = \frac{P \cdot a_R \, dv'}{4\pi\varepsilon R^2} \quad \text{——— (22)}$$

P → Dipole moment per unit volume

P dv' → Dipole moment for differential volume



The electric potential due to dipole,

$$V = \frac{P \cdot a_R}{4\pi\varepsilon R^2}$$

also, $P = Qd$

$d\cos\theta = d \cdot a_R$

ρ_{ps} → surface charge density

ρ_{pv} → volume charge density

potential, $$V = \int_S \frac{P \cdot a_n}{4\pi\varepsilon_0 R} ds' + \int_v \frac{-\nabla \cdot P}{4\pi\varepsilon_0 R} dv' \quad \text{——— (23)}$$

p = dipole moment per unit volume

so, we get,
$$\begin{cases} \rho_{ps} = P \cdot a_n \\ \rho_{pv} = -\nabla \cdot P \end{cases} \quad \text{—— (24)}$$

So, we can say that, where, the polarization occurs, an equivalent volume charge density ρ_{pv} is formed throughout the dielectric, while an equivalent surface charge density ρ_{ps} is formed over the surface of the dielectric.

We refer, ρ_{ps} and ρ_{pv} as bound (or polarization) surface and volume charge density respectively, as distinct from free surface and volume charge density ρ_s and ρ_v.

So, The total positive bound charge on Surface S bounding the dielectric is
$$Q_b = \oint P \cdot dS = \int \rho_{ps}\, dS \quad \text{—— (25)}$$

while the charge that remains inside S is
$$-Q_b = \int \rho_{pv}\, dv = -\int_v \nabla \cdot P\, dv \quad \text{—— (26)}$$

If the entire ~~dial~~ dielectric were electrically neutral prior to application of the electric field and if we have not added any free charge, the dielectric will remain electrically neutral.

so, the total charge $= \oint_S \rho_{ps}\, dS + \int_v \rho_{pv}\, dv = Q_b - Q_b = 0$

→ we consider the case in which the dielectric region contain free charge. If ρ_v is the volume density of free charge, the total volume charge density ρ_t is given by
$$\rho_t = \rho_v + \rho_{pv} = \nabla \cdot D = \nabla \cdot \epsilon_0 E \quad \text{—— (27)}$$

Hence, $\rho_v = \nabla \cdot \epsilon_0 E - \rho_{pv} = \nabla \cdot \epsilon_0 E - (-\nabla \cdot P)$

$$= \nabla \cdot (\epsilon_0 E + P)$$

$$= \nabla \cdot D$$

where, $\boxed{D = \epsilon_0 E + P}$ —— (28)

So, the net effect of the dielectric on the electric field E is to increase D inside by amount P.

so the application of E to the dielectric material causes the flux density to be greatest than it would be in free space.

So, for free space, $P = 0$

then, $D = \epsilon_0 E$

For some dielectrics, P is proportional to the applied electric field E, and we have,

$$\boxed{P = \chi_e \epsilon_0 E} \quad \text{—— (29)}$$

where χ_e, known as the electric susceptibility of the material, is more or less a measure of how susceptible (or sensitive) a given dielectric is to electric fields.

→ Dielectric Constant and Strength

So, $D = \epsilon_0 E + P$

$$= \epsilon_0 E + \chi_e \epsilon_0 E$$

$$= \epsilon_0 (1 + \chi_e) E = \epsilon_0 \epsilon_r E$$

or, $\boxed{D = \epsilon E}$ (30), where $\epsilon = \epsilon_0 \epsilon_r$ → permittivity of the dielectric

and, $\boxed{\epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0}}$ —— (31)

ϵ_0 → permittivity in free space

ϵ_r → Dielectric constant or Relative permittivity

→ ε_r and χ_e are dimensionless, whereas ε and ε_0 are in farads per meter.

↳ Dielectric Breakdown

When the electric field in a dielectric is sufficiently large, it begins to pull electrons completely out of the molecules, and the dielectric becomes conducting. That condition is Dielectric breakdown.

So, The dielectric strength is the max^m electric field that a dielectric can tolerate or withstand without electric breakdown.

↳ Continuity Equation and Relaxation Time

From the principal of charge conservation, the time rate of decrease of charge within a given volume must be equal to net outward current flow through the surface of the volume.

$$\text{So, } \left\{ I_{out} = \oint J\cdot ds = -\frac{dQ_{in}}{dt} \right\} \quad \text{——— (32)}$$

where Q_{in} → total charge enclosed by the closed surface.

using divergence theorem, $\oint_S J\cdot ds = \int_V (\nabla\cdot J)\, dv$

and, $Q_{in} = \int_V \rho_v\, dv$

Sm, using above equations,

$$\oint J\cdot ds = \int_V (\nabla\cdot J)\, dv = -\frac{dQ_{in}}{dt} = -\frac{d}{dt}\int \rho_v\, dv$$

$$\Rightarrow \int_V (\nabla\cdot J)\, dv = -\int_V \frac{\partial \rho_v}{\partial t}\, dv$$

$$\boxed{\nabla\cdot J = -\frac{\partial \rho_v}{\partial t}} \quad \text{——— (33)}$$

continuity equation

It is continuity of current equation or just continuity equation.
↳ derived from principal of conservation of charge.

for steady currents,

$$\frac{\partial \rho_v}{\partial t} = 0 \text{ and hence } \nabla . J = 0$$

it shows that total charge leaving a volume is same as the total charge entering it. (Kirchhoff's current law follow from this.)

→ If we introduce charge at some interior point of a given material like conductor

then, $J = \sigma E$ and using Gauss's law, $\nabla . D = \rho_v$

or, $\nabla . E = \frac{\rho_v}{\varepsilon}$

Now, we have $\nabla . J = -\frac{\partial \rho_v}{\partial t}$

$$\nabla . \sigma E = \sigma(\nabla . E) = \sigma \frac{\rho_v}{\varepsilon} = -\frac{\partial \rho_v}{\partial t}$$

or, $\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon} \rho_v = 0$ —— (32)

By seperating the variable in the differential eqn

$$\int \frac{\partial \rho_v}{\rho_v} = -\frac{\sigma}{\varepsilon} \partial t$$

By integrating both side

$$\ln \rho_v = -\frac{\sigma}{\varepsilon} t + \ln \rho_{v0}$$ (ρ_{v0} is constant of integration) is ρ_v at, $t = 0$

or $$\boxed{\rho_v = \rho_{v0} e^{-\sigma/\varepsilon t} = \rho_{v0} e^{-t/T_r}}$$ ←— (33)

where, $$\boxed{T_r = \frac{\varepsilon}{\sigma}}$$ → where T_r is the time constant in seconds it is known as relaxation time or rearrangement time.

—— (34)

Relaxation time is the time it takes a charge placed in the interior of a material to drop to e^{-1} ($= 36.8\%$) of its initial value

↳ It is short for good conductor and long for good dielectric

Ex For copper, $\sigma = 5.8 \times 10^{7}$ S/m , $\varepsilon_r = 1$

$$T_r = \frac{\varepsilon_r \varepsilon_0}{\sigma} = 1 \times \frac{10^{-9}}{36\pi} \times \frac{1}{5.8 \times 10^{7}} = 1.53 \times 10^{-19} \text{ sec}$$

Showing a rapid decay of charge placed inside copper.

This implies that for good conductor, the relaxation time is so short that most of the charge will vanish from any interior point and appear at the surface (as surface charge) almost instantaneously.

Ex for, quartz, $\sigma = 10^{-17}$ S/m , $\varepsilon_r = 5.0$

$$T_r = 5 \times \frac{10^{-9}}{36\pi} \times \frac{1}{10^{-17}} = 51.2 \text{ days}$$

Thus for good dielectrics, one may consider the introduced charge to remain wherever placed for times up to days.

⇒ BOUNDARY CONDITIONS

Till now we considered the existence of the electric field in a homogeneous medium.

If the field exists in a region consisting of two different media, then the condition that the field must satisfy at the interface seperating the media are called boundary condition.

To determine the boundary condition, we need to use Maxwells equations:

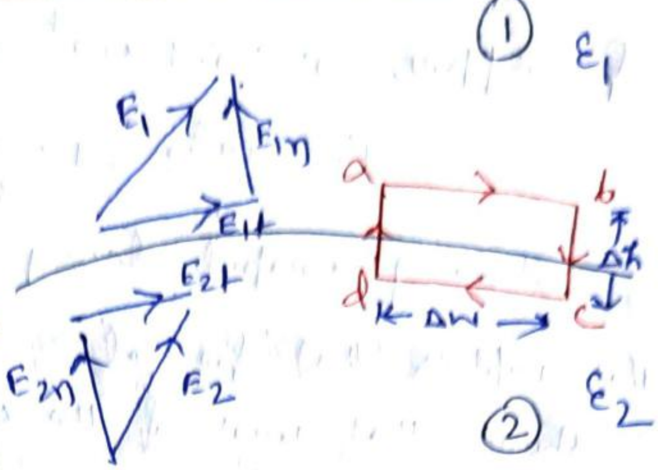
$$\oint E \cdot dl = 0 \quad \text{—— (35)}$$

and, $\oint_S D \cdot dS = Q_{enclosed}$ → free charge enclosed by surface S —— (36)

and, $E = E_t + E_n$ → E decompose into two orthogonal components, E_t & E_n

tangential Normal —— (37)

(ii) Dielectric - Dielectric Boundary Conditions.



two Dielectric region,

$\varepsilon_1 = \varepsilon_0 \varepsilon_{r_1}$

$\varepsilon_2 = \varepsilon_0 \varepsilon_{r_2}$

we can write, E in tangential & normal components

$E_1 = E_{1t} + E_{1n}$

$E_2 = E_{2t} + E_{2n}$

we apply one of the maxwell eqn $\oint E \cdot dl = 0$ in to the closed path, abcda, assuming that the path is very small w.r.to the spatial variation of E.

$$0 = E_{1t}\,\Delta w - \cancel{E_{1n}\frac{\Delta h}{2}} - \cancel{E_{2n}\frac{\Delta h}{2}} - E_{2t}\,\Delta w + \cancel{E_{2n}\frac{\Delta h}{2}} + \cancel{E_{1n}\frac{\Delta h}{2}}$$

E_{1n} & $\frac{\Delta h}{2}$ are opposite dir

E_{2n} & $\frac{\Delta h}{2}$ opposite dir

E_{2t} and, Δw are opposite dir

So, $(E_{1t} - E_{2t})\,\Delta w = 0$ because, $\frac{\Delta h}{2}$ terms cancel.

So, $\boxed{E_{1t} = E_{2t}}$ —— (38)

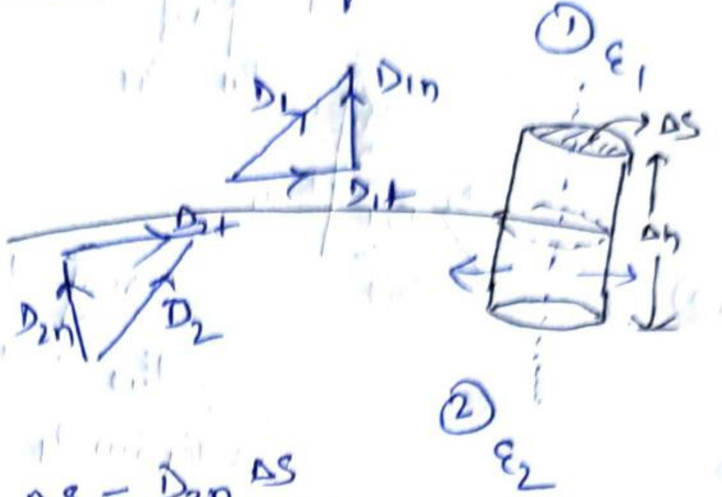
Thus the tangential components of E are the same on the two sides of the boundary.

i.e., E_t undergoes no change on the boundary and it is said to be continuous across the boundary.

since $D = \varepsilon E$

So, $\boxed{\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}}$ —— (39)

so, D_t undergoes some change across the interface. Hence D_t is said to be discontinuous across the interface. (15)



↳ Now, using another eqn

$$\oint_S D \cdot dS = Q_{enclosed}$$

Now, again the contribution due to side vanishes,

so, Allowing $\Delta h \to 0$ gives,

$$\Delta Q = \rho_s \Delta S = D_{1n}\,\Delta S - D_{2n}\,\Delta S$$

or, $\boxed{D_{1n} - D_{2n} = \rho_s}$ — (40)

where, ρ_s is free charge density at the surface (boundary)

if there is no free charge exist at the interface, then $\rho_s = 0$

so $\boxed{D_{1n} = D_{2n}}$ — (41)

Thus the normal component of D is continuous across the interface; so, D_n undergoes no change at the boundary

Since, $D = \varepsilon E$ so, $\boxed{\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}}$ — (42)

normal component of E is discontinuous at boundary.

So, The Boundary conditions are :-

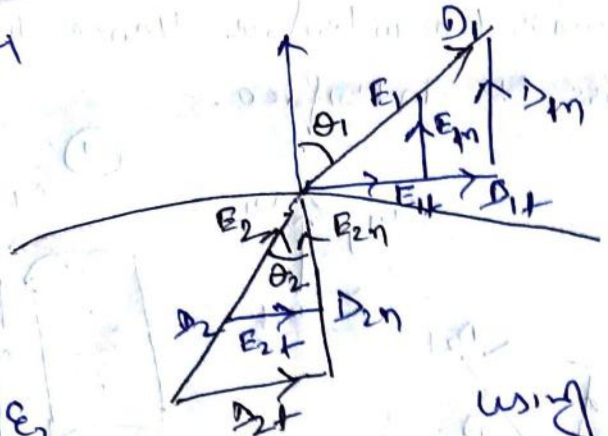
$$\begin{cases} E_{1t} = E_{2t} \\ D_{1n} - D_{2n} = \rho_s \\ D_{1n} = D_{2n} \quad \text{if } \rho_s = 0 \end{cases}$$ — (43)

↳ We can also use the boundary conditions to determine the "refraction" of the electric field across the interface

16

① ε_1

② ε_2



D_1 on E_1 and, D_2 on E_2 making an angles θ_1 and θ_2 with the normal to the interface.

using tangential Boundary condition,

$$E_{1t} = E_{2t}$$

$$\Rightarrow E_1 \sin\theta_1 = E_2 \sin\theta_2 \quad \text{—— (44)}$$

using normal boundary condition, (assuming $\rho_s = 0$), boundary is free of charge.

$$D_{1n} = D_{2n}$$

$$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

$$\Rightarrow \varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2 \quad \text{—— (45)}$$

using these two eqns

$$\frac{E_1 \sin\theta_1}{\varepsilon_1 E_1 \cos\theta_1} = \frac{E_2 \sin\theta_2}{\varepsilon_2 E_2 \cos\theta_2}$$

$$\Rightarrow \frac{\tan\theta_1}{\varepsilon_1} = \frac{\tan\theta_2}{\varepsilon_2} \quad \text{—— (46)}$$

Since, $\varepsilon_1 = \varepsilon_0 \varepsilon_{r_1}$ and, $\varepsilon_2 = \varepsilon_0 \varepsilon_{r_2}$

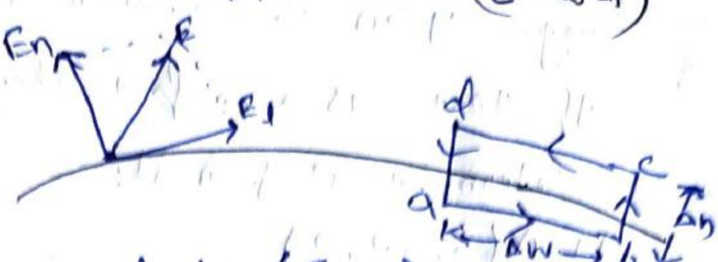
$$\Rightarrow \boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_{r_1}}{\varepsilon_{r_2}}} \quad \text{—— (47)}$$

Thus, an interface betn two dielectrics produces bending of the flux lines (ie, Electrical lines) as a result of unequal polarization charges the accumulate on the opposite side of the interface

→ Conductor – Dielectric Boundary Conditions

Dielectric ($\varepsilon = \varepsilon_0 \varepsilon_r$)

For perfect conductor, ($\sigma \to \infty$)



In closed path, abcda

$$\oint E \cdot dl = 0$$

conductor ($E = 0$)

$$0 = 0 \cdot \Delta W + 0 \cdot \frac{\Delta h}{2} + E_n \cdot \frac{\Delta h}{2} - E_t \cdot \Delta W - E_n \cdot \frac{\Delta h}{2} - 0 \cdot \frac{\Delta h}{2}$$

$$= E_n \frac{\Delta h}{2} - E_t \cdot \Delta W - E_n \cdot \frac{\Delta h}{2}$$

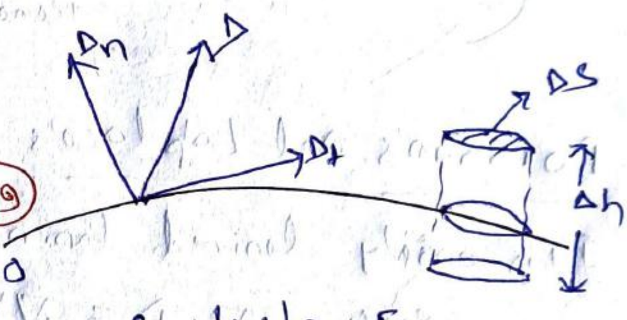
so, $\frac{\Delta h}{2}$ will be canceled
and, if we took $\Delta h \to 0$ (because, we have to check the condition only at interface)

then also, $\frac{\Delta h}{2}$ term cancel out,

so, $E_t \cdot \Delta W = 0$

$$\boxed{E_t = 0} \quad (48)$$

or, $\boxed{D_t = \varepsilon_0 \varepsilon_r E_t = 0}$ (49)



→ using $\oint_S D \cdot dS = Q_{enclosed}$
and, letting $\Delta h \to 0$

Conductor, $E = 0$

$\Delta Q = D_n \cdot \Delta S - D \cdot \Delta S$

so, $D = \varepsilon E = 0$, inside the conductor,

so, $D_n = \frac{\Delta Q}{\Delta S} = \rho_s$ or, $\boxed{D_n = \rho_s}$ (50) → $\boxed{D_n = \varepsilon_0 \varepsilon_r E_n = \rho_s}$ (51)

→ Conductor – Free Space Boundary Conditions

it is the special case of conductor – dielectric conditions.

$D_t = \varepsilon_0 \varepsilon_r E_t = 0$ and, $D_n = \varepsilon_0 \varepsilon_r E_n = \rho_s$

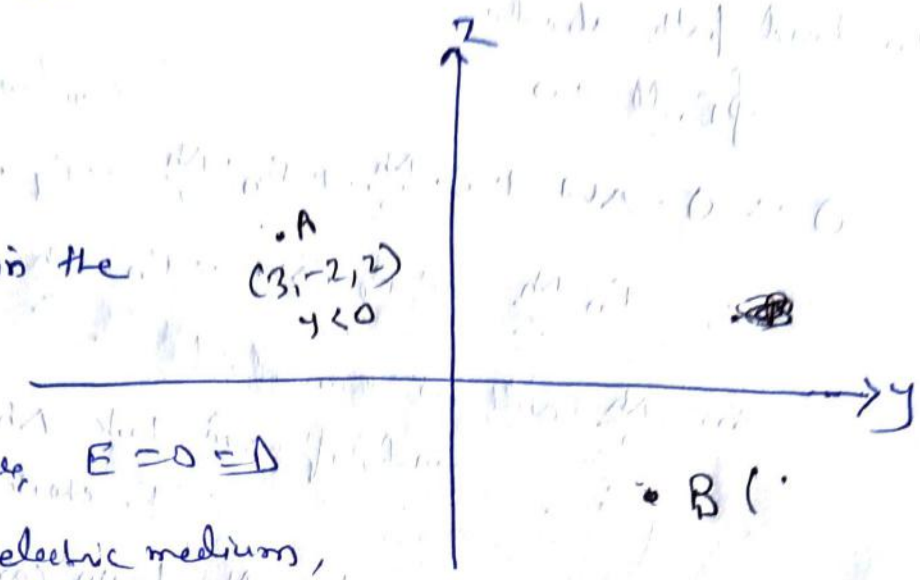
here $\varepsilon_r = 1$, for free space $\quad$ $\varepsilon_r = 1$ for free space

so, $\boxed{D_t = \varepsilon_0 E_t = 0}$ (52) $\quad$ so, $\boxed{D_n = \varepsilon_0 E_n = \rho_s}$ (53)

(18)

Q Region, $y < 0$ consists of a perfect conductor while region $y > 0$ is dielectric medium $(\varepsilon_r = 2)$ as in figure. If there is a surface charge of $2\ nC/m^2$ on the conductor, determine E & D at:

(a) $A(3, -2, 2)$
(b) $B(-4, 1, 5)$



(a) $A(3, -2, 2)$ is in the conductor, since, $y = -2 < 0$ Hence, $E = 0 = D$

(b) $B(-4, 1, 5)$ is in dielectric medium,
Since, $y = 1 > 0$ So, $D_n = \rho_s = 2\ nC/m^2$

Hence, $E = \dfrac{D}{\varepsilon_0 \varepsilon_r} = 2\times10^{-9} \times \dfrac{36\pi \times 10^{9}}{2}$
$= 36\pi\, a_y\ V/m$

↳ Poisson's and Laplace's Equations

It is easily derived from Gauss's law

Since, $\nabla \cdot D = \nabla \cdot \varepsilon E = \rho_v$

and $E = -\nabla V$

So $\nabla \cdot (-\varepsilon \nabla V) = \rho_v \Rightarrow \boxed{\nabla^2 V = -\dfrac{\rho_v}{\varepsilon}}$ — (54)

It is known as Poisson's Equation, and for the special case, when $(\rho_v = 0)$, i.e. for charge free region

$\boxed{\nabla^2 V = 0}$ → known as Laplace's equation — (55)

In cartesian, $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0$

In cylindrical, $\dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\rho \dfrac{\partial V}{\partial \rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 V}{\partial \phi^2} + \dfrac{\partial^2 V}{\partial z^2} = 0$ — (56)

& In spherical, $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial V}{\partial r}\right) + \dfrac{1}{r^2 \sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta \dfrac{\partial V}{\partial \theta}\right) + \dfrac{1}{r^2 \sin^2\theta}\dfrac{\partial^2 V}{\partial \phi^2} = 0$

⇒ RESISTANCE AND CAPACITANCE.

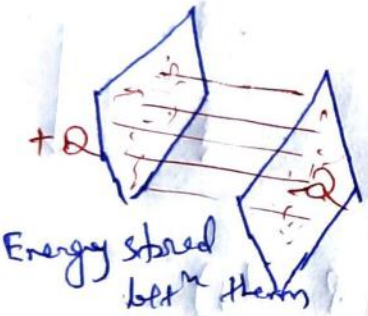
In general we consider resistance of a conductor of uniform cross section. If the cross-section of conductor is not uniform then the resistance is obtained from

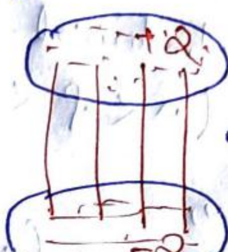
$$R = \frac{V}{I} = \frac{\int E \cdot dl}{\oint \sigma E \cdot ds}$$

CAPACITOR

It is a device used to store electrostatic energy in the form of electric field lines.

→ It is a two conductor system seperated by dielectric



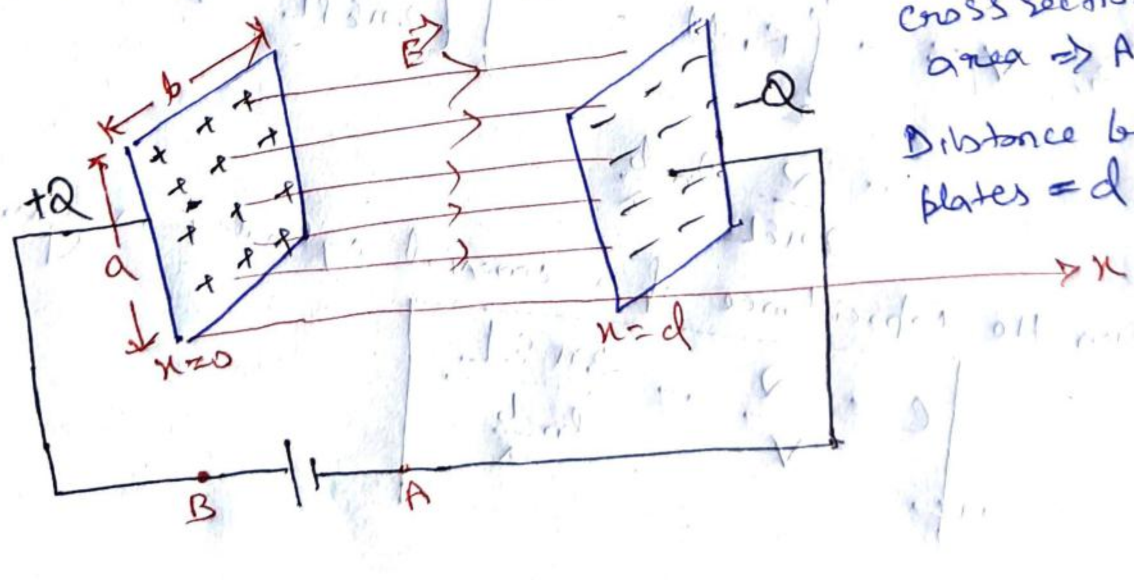


energy stored bet^n them

→ Parallel plate Capacitor.

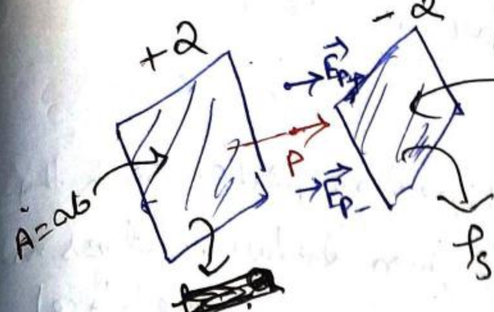
→ 2 finite parallel rectangular plates seperated by very small distance $(A >>> d)$

→ It means the area is as much compare to distance bet^n them that the finite plate should behave as infinite plate.



Cross section area ⇒ $A = ab$

Distance bet^n plates $= d$



$\rho_s = \frac{Q}{ab}$ $\quad$ $\rho_s = \frac{Q}{ab}$

$$\vec{E}_{P_+} = \frac{\rho_s}{2\epsilon}\hat{a}_n = \frac{Q}{2ab\epsilon}\hat{a}_n$$

$$\vec{E}_{P_-} = \frac{\rho_s}{2\epsilon}\hat{a}_n = \frac{Q}{2ab\epsilon}\hat{a}_n$$

So, $$\boxed{\vec{E}_P = \vec{E}_{P_+} + \vec{E}_{P_-} = \frac{Q}{ab\epsilon}\hat{a}_n}$$

So, $\vec{E}$ is dependent on Area, Q and ϵ and does not depend on distance.

→ Potential difference betn 2 plates

$$V_{AB} = V_B - V_A = -\int_A^B \vec{E}\cdot dl$$

$$V_+ - V_- = -\int_-^+ \vec{E}\cdot d\vec{l} = -\int_-^+ \left(\frac{\rho_s}{\epsilon}\hat{a}_n\right)\cdot dx\,\hat{a}_n$$

Since, +, at '+' x = 0
-, at '-' x = d

So, $$V_{AB} = V_+ - V_- = -\int_{x=d}^{x=0} \frac{\rho_s}{\epsilon}\,dx$$

So, $$\boxed{V_{AB} = V_+ - V_- = \frac{\rho_s}{\epsilon}d} = E\cdot d$$

So, $$E = \frac{V_{AB}}{d} = \frac{V_+ - V_-}{d}$$

→ Capacitance of parallel plate capacitor

$$C = \frac{Q}{(V_+ - V_-)} = \frac{\text{magnitude of charge on any of the conductor}}{\text{potential difference betn plate}}$$

$$\boxed{C = \frac{Q}{E\cdot d} = \frac{\rho_s A}{\frac{\rho_s}{\epsilon}d} = \frac{A\epsilon}{d} = \frac{(ab)\epsilon}{d}}$$

→ Force of one parallel plate due to the uniformly charged another plate.

$\vec{E} = \frac{\rho_s}{2\epsilon}\hat{a}_n$ due to single plate

$$\vec{F} = +Q\vec{E}_{ext}$$

$$d\vec{F} = dQ\,\vec{E}_{ext}$$

$$\boxed{\vec{F} = \frac{Q\rho_s}{2\epsilon}\hat{a}_n = \frac{Q^2}{2ab\epsilon}\hat{a}_n}$$

→ Breakdown voltage of parallel plate capacitor.

$V \rightarrow V_{max}$ (dielectric breakdown voltage)

↓

Q_{max} (max^m amount of charge stored on plate such that dielectric breakdown of given dielectric does not take place)

$\vec{E}_{max}$ (dielectric strength of parallel plate capacitor)

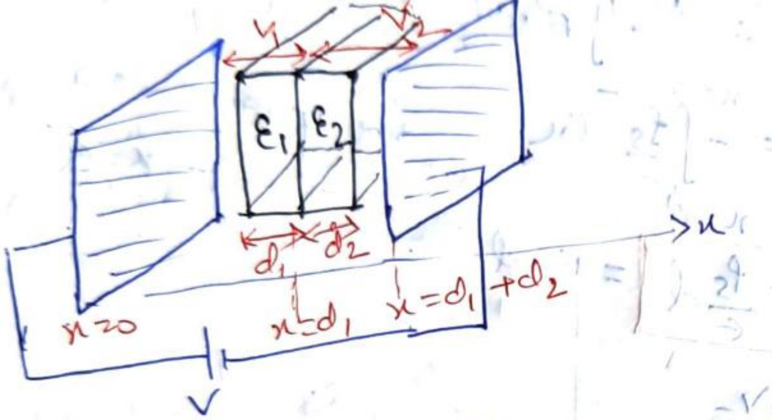
$$\boxed{\vec{E}_{max} = \frac{Q_{max}}{ab\epsilon} = \frac{V_{max}}{d}}$$

If, $V > V_{max} \longrightarrow Q > Q_{max}$

↓

Dielectric Breakdown $\leftarrow E > E_{max}$

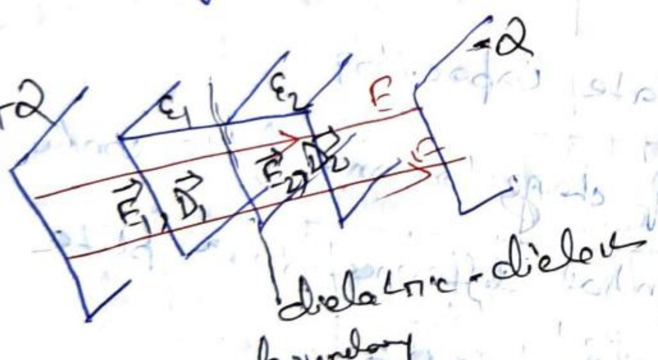
→ Series Combination of parallel plate capacitor



Equivalent ckt

$V = V_1 + V_2$

$$\frac{Q}{C} = \frac{Q}{C_1} + \frac{Q}{C_2} \Rightarrow \left\{ \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \right\}$$



Using boundary condition

$\vec{E}_1 = \vec{E}_{1n}$

$\vec{E}_2 = \vec{E}_{2n}$

$\vec{D}_1 = \vec{D}_{1n}$

$\vec{D}_2 = \vec{D}_{2n}$

Since there is no tangential component, only normal component present.

Using boundary condition

(i) $\vec{E}_{1t} = \vec{E}_{2t} = 0$

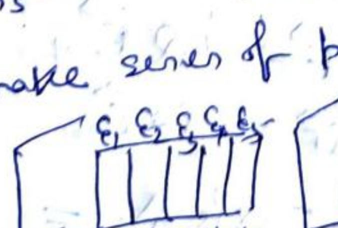
(ii) $\vec{D}_{1n} = \vec{D}_{2n}$ or, $\vec{D}_1 = \vec{D}_2$

since, $\rho_s = 0$ at boundary

but, $\epsilon_1 E_1 = \epsilon_2 E_2$

so, $\boxed{\frac{E_1}{E_2} = \frac{\epsilon_2}{\epsilon_1} \neq 1}$

→ So, if we make series of parallel plate capacitor,



Here, $D_1 = D_2 = D_3 = D_4 = D_5$

; $E_1 \neq E_2 \neq E_3 \neq E_4 \neq E_5$

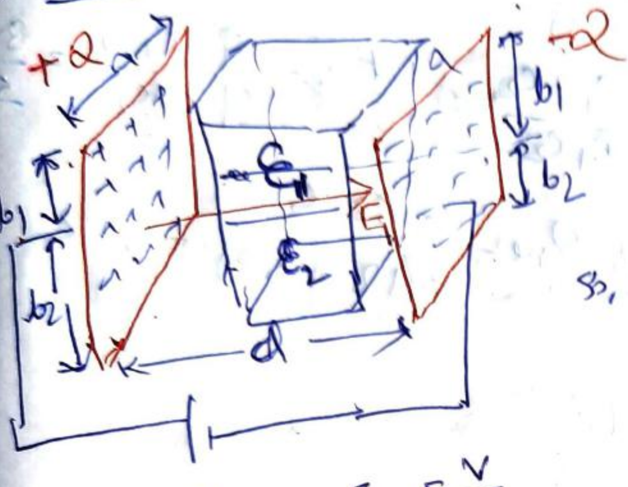
Since, we calculate, $\vec{E} = \frac{Q}{ab\epsilon}$ so, for $E_1 = \frac{Q_1}{ab\epsilon_1}$, $E_2 = \frac{Q_2}{ab\epsilon_2}$

$D_1 = \frac{Q_1}{ab}$, $D_2 = \frac{Q_2}{ab}$

But since, $D_1 = D_2$

so, $\frac{Q_1}{ab} = \frac{Q_2}{ab} \Rightarrow \boxed{Q_1 = Q_2}$

→ Parallel Combination of parallel plate capacitor



(iii) $\epsilon_1 \neq \epsilon_2$

So, two capacitor with different ε we make,

so, $C_1 = \frac{A_1 \epsilon_1}{d} = \frac{(ab_1)\epsilon_1}{d}$

$C_2 = \frac{(ab_2)\epsilon_2}{d}$

$E_1 = \frac{V}{d}$, $E_2 = \frac{V}{d}$

so $\boxed{E_1 = E_2}$

so $\frac{D_1}{\epsilon_1} = \frac{D_2}{\epsilon_2} \Rightarrow \frac{D_1}{D_2} = \frac{\epsilon_1}{\epsilon_2} \neq 1$

so $\boxed{D_1 \neq D_2}$

Since, $E_1 = \frac{\rho_s}{\epsilon_1} = \frac{Q_1}{(ab_1)\epsilon_1}$

$E_2 = \frac{\rho_s}{\epsilon_2} = \frac{Q_2}{(ab_2)\epsilon_2}$

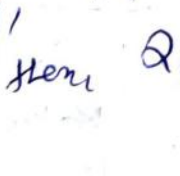
Since, $E_1 = E_2$

so, $\frac{Q_1}{ab_1\epsilon_1} = \frac{Q_2}{ab_2\epsilon_2} \Rightarrow \left\{ \frac{Q_1}{b_1\epsilon_1} = \frac{Q_2}{b_2\epsilon_2} \right\}$

so if, $b_1\epsilon_1 \neq b_2\epsilon_2$

then $Q_1 \neq Q_2$

so changes on parallel plate capacitor will not be same if $b_1\epsilon_1 \neq b_2\epsilon_2$

→ 

from Boundary condition, $\vec{E}_{1t} = \vec{E}_{2t}$ & $\vec{D}_{1n} = \vec{D}_{2n}$

$\vec{E_1} = \vec{E}_{1t} + \vec{E}_{1n}$, here, only tangential component exist

so $\vec{E_1} = \vec{E}_{1t} = \vec{E}_{2t} = \vec{E_2}$

so $\boxed{\vec{E_1} = \vec{E_2}}$ & $\boxed{\vec{D_1} \neq \vec{D_2}}$

⇒ CO-Axial Capacitor.

consider the length 'L' of two coaxial conductors of inner radius 'a' and outer radius 'b' (b > a).

The space bet^n the conductors be filled with a homogenous dielectric with permitivity ε. Assume that conductors 1 and 2, respectively carry +Q and −Q uniformly distributed on them.

→ By applying Gauss's law to an arbitrary Gaussian cylindrical surface of radius ρ $(a < \rho < b)$

from Gauss's law,

$$Q = \oint D \cdot ds = \epsilon \oint \vec{E} \cdot d\vec{s}$$

since $\vec{E} = E_\rho$ (along ρ only)

So $Q = \epsilon E_\rho \oint ds$

$= \epsilon E_\rho \cdot 2\pi \rho l \Rightarrow E_\rho = \dfrac{Q}{2\pi \epsilon \rho L}$

$$\Rightarrow E = \frac{Q}{2\pi \epsilon \rho L} \hat{a}_\rho$$

So, $V = -\int_2^1 E \cdot dl = -\int_b^a \left[\dfrac{Q}{2\pi \epsilon \rho L} \hat{a}_\rho\right] \cdot d\rho\, a_\rho$

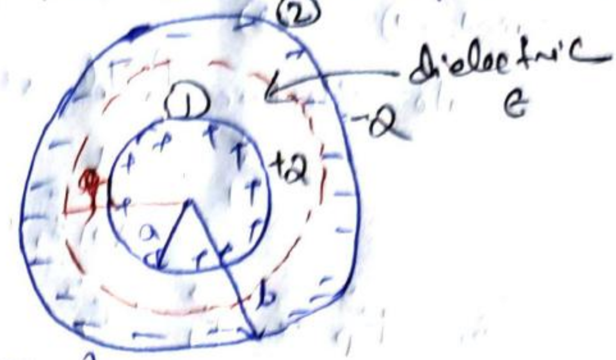
$$= \frac{Q}{2\pi \epsilon L} \ln \frac{b}{a}$$

Thus the capacitance of a coaxial cylinder is given by

$$\boxed{C = \frac{Q}{V} = \frac{2\pi \epsilon L}{\ln \frac{b}{a}}}$$

Spherical Capacitor

A spherical capacitor is the case of two concentric spherical conductors.



By applying Gauss's law to an arbitrary Gaussian spherical surface of radius $r (a < r < b)$

we have, $Q = \varepsilon \oint E \cdot ds = \varepsilon E_r 4\pi r^2$

$$\text{So } E = \frac{Q}{4\pi\varepsilon r^2}\hat{a}_r$$

The potential difference betn the conductors is

$$V = -\int_2^1 E \cdot dl = -\int_b^a \left[\frac{Q}{4\pi\varepsilon r^2}\hat{a}_r\right] \cdot dr\,\hat{a}_r$$

$$= \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]$$

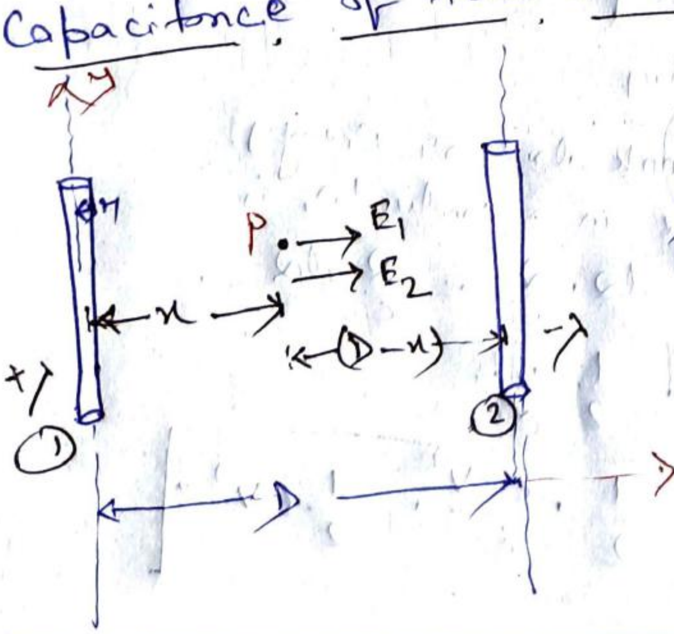
Thus, the capacitance of spherical capacitor is

$$\boxed{C = \frac{Q}{V} = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}}$$

by letting $b \to \infty$, $\boxed{C = 4\pi\varepsilon a}$ which is the capacitance of a spherical capacitor where outer plate is infinite large.

Also called isolated sphere.

=> Capacitance of two wire line



-> Consider the two cylindrical infinite wire line having radius 'r'.

-> The Distance betn two wire is D. $(D >>> r)$

-> The infinite wire line having uniform line charge density $+\lambda$ & $-\lambda$ respectively for wire ① & wire ②.

If we taking a arbitrary point p in between the two wire, (1) + (2)

The Electric field at point P due to (1) + (2) is,

$$\vec{E_1} = \frac{\lambda}{2\pi\varepsilon_0 x}\hat{a_x} \quad \& \quad \vec{E_2} = \frac{\lambda}{2\pi\varepsilon_0(D-x)}\hat{a_x}$$

So, $\vec{E_p} = \vec{E_1} + \vec{E_2}$

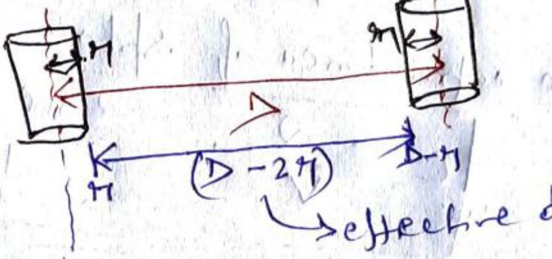
$$= \frac{\lambda}{2\pi\varepsilon_0 x}\hat{a_x} + \frac{\lambda}{2\pi\varepsilon_0(D-x)}\hat{a_x}$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\frac{1}{x} - \frac{1}{D-x}\right]\hat{a_x}$$

So Now, the potential betn (1) + (2)

$$V_{12} = -\int_{D-r}^{r}\vec{E}\cdot d\vec{x} = -\int_{D-r}^{r}\frac{\lambda}{2\pi\varepsilon_0}\left[\frac{1}{x} - \frac{1}{D-x}\right]dx$$

Key



$$V = \frac{\lambda}{2\pi\varepsilon}\left[2\ln\left(\frac{D-r}{r}\right)\right]$$

$V = \frac{\lambda}{\pi\varepsilon}\ln\frac{D}{r}$ Since, $D >> r$, $D + r \Rightarrow D$

So, the capacitance betn them (two wire)

$$\boxed{C = \frac{\lambda}{V} = \frac{\pi\varepsilon}{\ln(D/r)}}$$

→ Energy stored in a conductor

Since, the work done or Energy in a conductor, is for dQ charge

$$dW = dQ\,V = dQ\cdot\frac{Q}{C} = \frac{Q}{C}dQ$$

$$W = \int\frac{Q}{C}dQ = \frac{1}{2}\frac{Q^2}{C}$$

By put, $Q = CV$, $\boxed{W = \frac{1}{2}\cdot\frac{C^2V^2}{C} = \frac{1}{2}CV^2}$

ELECTROSTATIC FIELDS

COULOMB'S LAW AND FIELD INTENSITY

Coulomb's law states that the force F betn two point charges Q_1 and Q_2 is:

(i) Along the line joining them

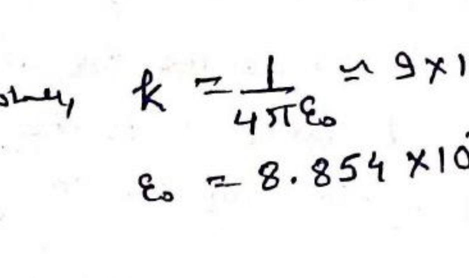
(ii) Directly proportional to the product $Q_1 Q_2$ of the charge

(iii) Inversely proportional to the square of the distance R betn them.

so, mathematically, $$\left\{ F = \frac{kQ_1Q_2}{R^2} \right\} \quad \text{—— (1)}$$

where, $k = \frac{1}{4\pi\varepsilon_0} \simeq 9\times10^{9}$ m/F

$\varepsilon_0 = 8.854\times10^{-12} = \frac{10^{-9}}{36\pi}$ F/m (Farad/meter)



r_1 & r_2 are position vector

F_{12} = force F_{12} on Q_2 due to Q_1

$$\left\{ F_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2} a_{R_{12}} \right\} \quad \text{—— (2)}$$

where, $R_{12} = r_2 - r_1$

$R = |R_{12}| = |r_2 - r_1|$

$a_{R_{12}} = \frac{R_{12}}{R}$ → unit vector along R_{12}

or, $$\left\{ F_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 |r_2 - r_1|^2} \cdot \frac{(r_2 - r_1)}{|r_2 - r_1|} = \frac{Q_1 Q_2 (r_2 - r_1)}{4\pi\varepsilon_0 |r_2 - r_1|^3} \right\} \quad \text{—— (3)}$$

important points,

(i) $F_{21} = |F_{12}| a_{R_{21}} = |F_{12}|(-a_{R_{12}})$

or, $F_{21} = -F_{12}$

(ii) Q_1 and Q_2 must be static

(iii) The signs of Q_1 and Q_2 must be taken into account for like charges $Q_1 Q_2 > 0$. For unlike charges $Q_1 Q_2 < 0$.

→ If we have more than two point charges, we can use the principle of superposition to determine the force on a particular charge.

The principle states that if there are N charges $Q_1, Q_2, \ldots, Q_N$ located respectively, at points with position vectors $r_1, r_2, \ldots, r_N$, the resultant force F on a charge Q located at point r is the vector sum of the forces exerted on Q by each of the charges $Q_1, Q_2, \ldots, Q_N$. Hence:

$$F = \frac{Q Q_1 (r - r_1)}{4\pi\varepsilon_0 |r - r_1|^3} + \frac{Q Q_2 (r - r_2)}{4\pi\varepsilon_0 |r - r_2|^3} + \cdots + \frac{Q Q_N (r - r_N)}{4\pi\varepsilon_0 |r - r_N|^3}$$

or,

$$F = \frac{Q}{4\pi\varepsilon_0} \sum_{k=1}^{N} \frac{Q_k (r - r_k)}{|r - r_k|^3} \quad \text{——— (4)}$$

→ Electric field intensity

The Electric field intensity (or electric field strength) E is the force per unit charge when placed in an electric field.

$$E = \frac{F}{Q} \quad \text{——— (5A)}$$

$$E = \frac{Q}{4\pi \varepsilon_0 R^2} a_R = \frac{Q(r - r')}{4\pi \varepsilon_0 |r - r'|^3}$$

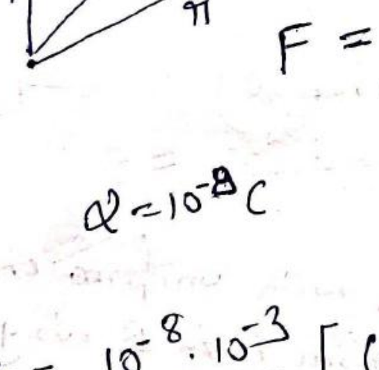
Similarly, for N point charges,

$$E = \frac{Q_1(r - r_1)}{4\pi \varepsilon_0 |r - r_1|^3} + \frac{Q_2(r - r_2)}{4\pi \varepsilon_0 |r - r_2|^3} + \cdots + \frac{Q_N(r - r_N)}{4\pi \varepsilon_0 |r - r_N|^3}$$

or,

$$E = \frac{1}{4\pi \varepsilon_0} \sum_{k=1}^{N} \frac{Q_k (r - r_k)}{|r - r_k|^3} \qquad \text{— 5(B)}$$

Q. point charges 1mC and −2mC are located at (3, 2, −1) and, (−1, −1, 4) respectively. calculate the electric force on a 10 nC charge located at (0, 3, 1) and the electric field intensity at the point.



$$F = \sum_{k=1,2} \frac{Q Q_k}{4\pi \varepsilon_0 R^2} a_R = \sum_{k=1,2} \frac{Q Q_k (r - r_k)}{4\pi \varepsilon_0 |r - r_k|^3}$$

$$F = \frac{Q}{4\pi\varepsilon_0} \left\{ \frac{10^{-3}\left[(0,3,1) - (3,2,-1)\right]}{|(0,3,1) - (3,2,-1)|^3} - \frac{2 \cdot 10^{-3}\left[(0,3,1) - (-1,-1,4)\right]}{|(0,3,1) - (-1,-1,4)|^3} \right\}$$

$Q = 10^{-8}$ C

$$F = \frac{10^{-8} \cdot 10^{-3}}{4\pi \cdot \frac{10^{-9}}{36\pi}} \left[\frac{(-3,1,2)}{(9+1+4)^{3/2}} - \frac{2(1,4,-3)}{(1+16+9)^{3/2}} \right]$$

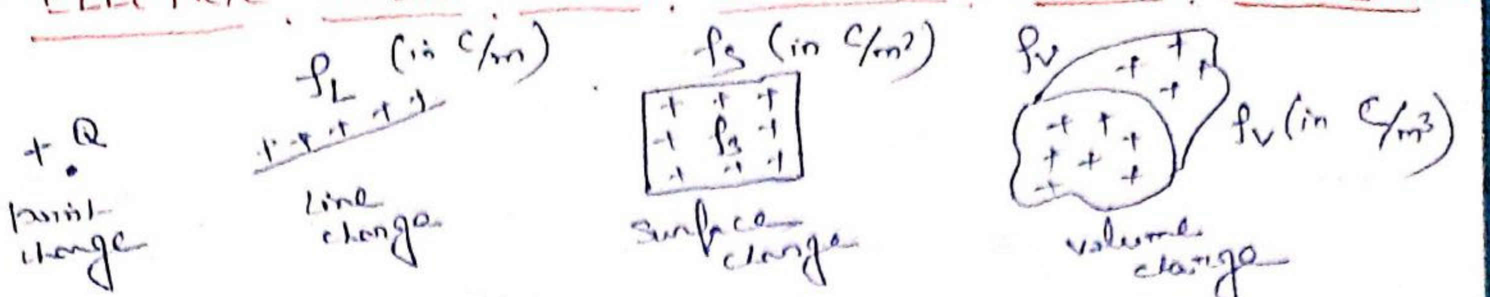
$$= 9 \cdot 10^{-2} \left[\frac{(-3,1,2)}{14\sqrt{14}} + \frac{(-2,-8,6)}{26\sqrt{26}} \right]$$

$$F = -6.507\, a_x - 3.817\, a_y + 7.506\, a_z \text{ mN}$$

At that point, $E = F/Q = (-6.507, -3.817, 7.506) \cdot \frac{10^{-3}}{10 \times 10^{-9}}$

$$= -650.7\, a_x - 381.7\, a_y + 750.6\, a_z \ \frac{\text{kV}}{\text{m}}$$

ELECTRIC FIELDS DUE TO CONTINOUS CHARGE DISTRIBUTION



The charge ~~distribution~~ element dQ and the total charge Q due to these charge distribution are obtained as,

$$dQ = \rho_L \, dl \rightarrow Q = \int_L \rho_L \, dl \quad \text{(line charge)}$$

$$dQ = \rho_S \, dS \rightarrow Q = \int_S \rho_S \, dS \quad \text{(surface charge)}$$

$$dQ = \rho_V \, dv \rightarrow Q = \int_V \rho_V \, dv \quad \text{(volume charge)}$$

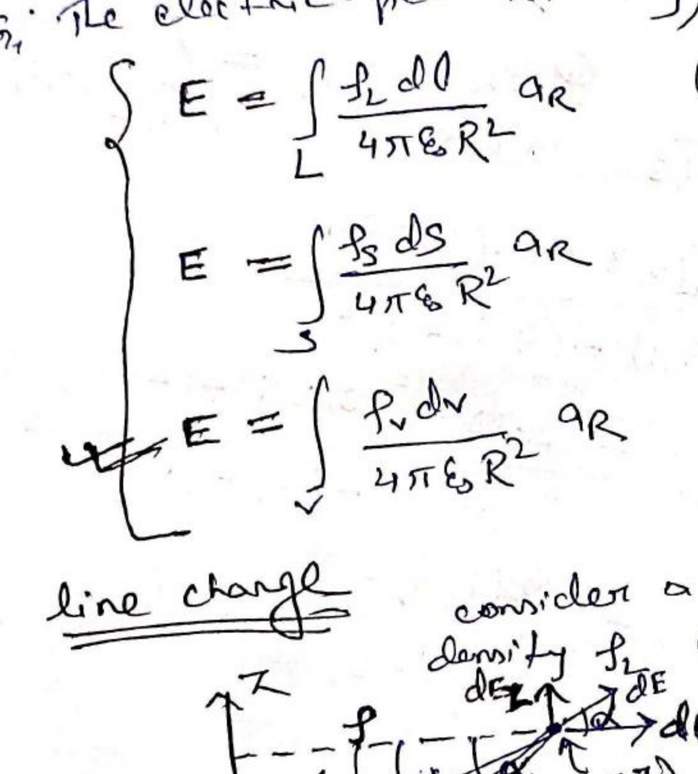
So, The electric field intensity, by replacing Q

$$E = \int_L \frac{\rho_L \, dl}{4\pi\varepsilon_0 R^2} a_R \quad \text{(line charge)} \quad \text{--- (6)}$$

$$E = \int_S \frac{\rho_S \, dS}{4\pi\varepsilon_0 R^2} a_R \quad \text{(surface charge)} \quad \text{--- (7)}$$

$$E = \int_V \frac{\rho_V \, dv}{4\pi\varepsilon_0 R^2} a_R \quad \text{(volume charge)} \quad \text{--- (8)}$$

→ **line charge** consider a line charge with uniform charge density ρ_L extending from A to B along z-axis.



$$dQ = \rho_L \, dl = \rho_L \, dz'$$

$$Q = \int_{z_A}^{z_B} \rho_L \, dz' \quad \text{--- (9)}$$

the total charge

The position vector,

$$R = (x, y, z) - (0, 0, z') \quad \text{--- (10)}$$
$$= x\,a_x + y\,a_y + (z - z')\,a_z$$

or $R = \rho a_\rho + (z - z') a_z$ Since, $\rho a_\rho = x a_x + y a_y$

$$|R|^2 = x^2 + y^2 + (z - z')^2 \quad \text{--- (11)}$$
$$= \rho^2 + (z - z')^2$$

So, $\dfrac{a_R}{|R|^2} = \dfrac{\text{unit vector along position vector } R}{(\text{position vector } R \text{ magnitude})^2} = \dfrac{R}{|R| \cdot |R|^2} = \dfrac{R}{|R|^3}$

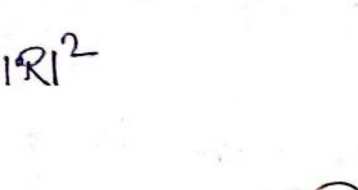
So $\dfrac{a_R}{|R|^2} = \dfrac{\rho a_\rho + (z - z') a_z}{[\rho^2 + (z - z')^2]^{3/2}}$ --- (12)

Substituting all the above in the eqn (6) of line charge

$$E = \int \frac{\rho_L \, dl'}{4\pi \varepsilon_0 R^2} a_R \qquad \text{where, } R^2 = |R|^2 \quad \text{--- (13)}$$

So, $E = \dfrac{\rho_L}{4\pi \varepsilon_0} \displaystyle\int \frac{\rho a_\rho + (z - z') a_z}{[\rho^2 + (z - z')^2]^{3/2}} dz'$

to evaluate this, it is convenient that we integrate along α_1 to α_2 instead along z.



$\cos\alpha = \dfrac{\rho}{|R|}$

$\rho = |R| \cos\alpha$

$z - z' = |R| \sin\alpha$ --- (16)

$R = [\rho^2 + (z - z')^2]^{1/2} = \rho \sec\alpha$ --- (14)

$\cos\alpha = \dfrac{\rho}{R} \Rightarrow R = \dfrac{\rho}{\cos\alpha}$

$\Rightarrow R = \rho \sec\alpha$

$\tan\alpha = \dfrac{z - z'}{\rho}$

$z - z' = \rho \tan\alpha$

So, $z - z' = \rho \tan\alpha$

or, $z' = z - \rho \tan\alpha$

So, $\dfrac{dz'}{d\alpha} = 0 - \rho \sec^2\alpha$

$\Rightarrow dz' = -\rho \sec^2\alpha \, d\alpha$ --- (15)

So, eqⁿ (13) becomes

$$E = \frac{\rho_L}{4\pi\varepsilon_0}\int \frac{\rho\, a_\rho + (z-z')a_z}{[\rho^2 + (z-z')^2]^{3/2}}\, dz'$$

using the eqⁿ 14, 15, 16

$$= \frac{\rho_L}{4\pi\varepsilon_0}\int \frac{|R|\cos\alpha\, a_\rho + |R|\sin\alpha\, a_z}{|R|^3}\, dz'$$

$$= \frac{\rho_L}{4\pi\varepsilon_0}\int \frac{\cos\alpha\, a_\rho + \sin\alpha\, a_z}{|R|^2}\, dz'$$

change the limits from z to α

$$= \frac{\rho_L}{4\pi\varepsilon_0}\int_{\alpha_1}^{\alpha_2} \frac{\cos\alpha\, a_\rho + \sin\alpha\, a_z}{\rho^2 \sec^2\alpha}\cdot(-\rho\sec^2\alpha\, d\alpha)$$

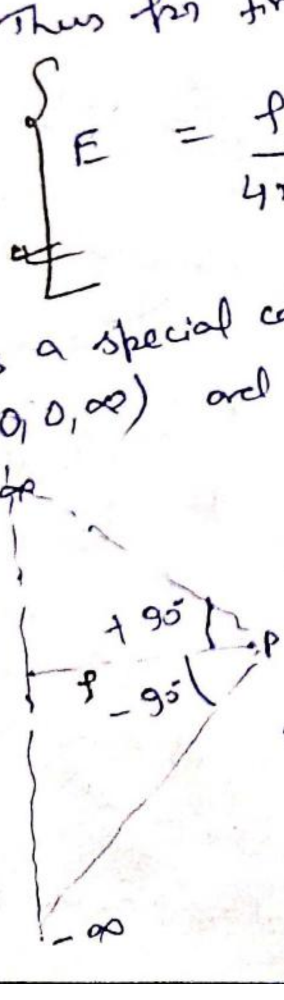
$$= -\frac{\rho_L}{4\pi\varepsilon_0\rho}\int_{\alpha_1}^{\alpha_2}[\cos\alpha\, a_\rho + \sin\alpha\, a_z]\, d\alpha$$

Thus for finite charge

$$E = \frac{\rho_L}{4\pi\varepsilon_0\rho}\left[-(\sin\alpha_2 - \sin\alpha_1)a_\rho + (\cos\alpha_2 - \cos\alpha_1)a_z\right] \quad \text{—— (17)}$$

As a special case, for an infinite line charge, point B is at (0, 0, ∞) and A is at (0, 0, −∞)

so that $\alpha_1 = \pi/2$ and, $\alpha_2 = -\pi/2$



so $E = \frac{\rho_L}{4\pi\varepsilon_0\rho}\left[-(-1-1)a_\rho + (0-0)a_z\right]$

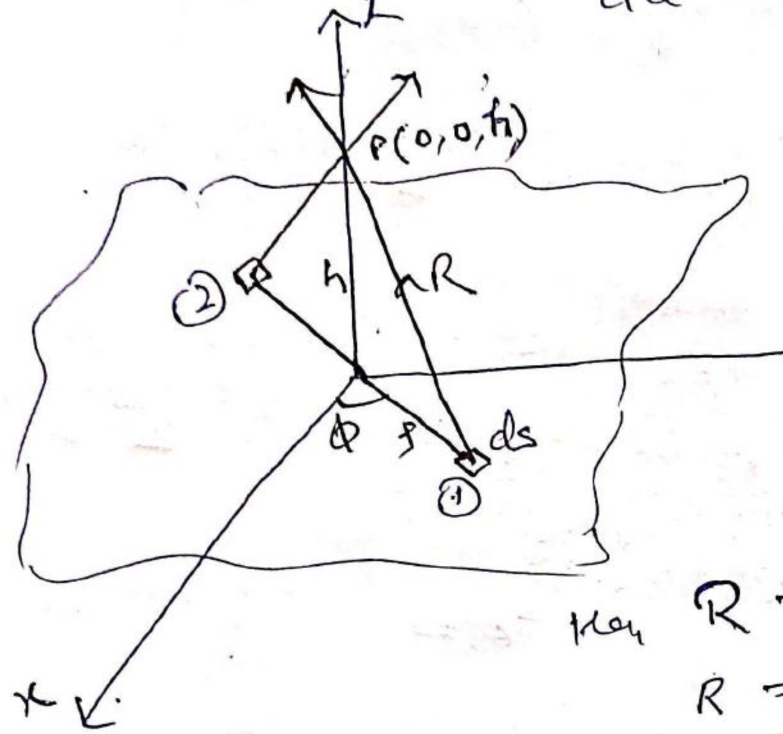
So,

$$\boxed{E = \frac{\rho_L}{4\pi\varepsilon_0\rho}a_\rho} \quad \text{—— (18)}$$

→ Surface charge

Consider an infinite sheet of charge in the xy-plane with uniform charge density ρ_s. The charge associated with elemental area dS is

$$dQ = \rho_s \, dS$$



So, from eqn (7) the contribution of E field at point P(0,0,h) by the elemental surface (1)

$$dE = \frac{\rho_s \, dS}{4\pi\varepsilon_0 R^2} a_R \qquad \text{— (19)}$$

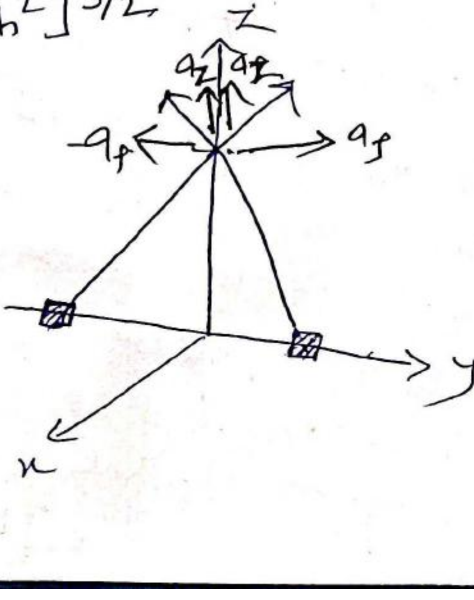
$$\left.\begin{aligned} \text{then } \quad \mathbf{R} &= \rho(-a_\rho) + h a_z \\ R = |\mathbf{R}| &= [\rho^2 + h^2]^{1/2} \\ a_R &= \frac{\mathbf{R}}{R} = \frac{\mathbf{R}}{|\mathbf{R}|} \\ \text{and } \quad dQ = \rho_s \, dS &= \rho_s \, \rho \, d\phi \, d\rho \end{aligned}\right\} \qquad \text{— (20)}$$

So, using (20) in (19)

$$dE = \frac{\rho_s \, \rho \, d\phi \, d\rho}{4\pi\varepsilon_0 R^2} a_R = \frac{\rho_s \, \rho \, d\phi \, d\rho}{4\pi\varepsilon_0} \cdot \frac{\mathbf{R}}{|\mathbf{R}|^3}$$

$$\int dE = \frac{\rho_s \, \rho \, d\phi \, d\rho \, [-\rho a_\rho + h a_z]}{4\pi\varepsilon_0 [\rho^2 + h^2]^{3/2}} \qquad \text{— (21)}$$

→ Since due to symmetry of charge distribution, for every element 1, there is a corresponding element 2, whose contribution along a_ρ cancels that of element 1.



~~Thus the contributions along a_ρ cancel that~~ (8.)

Thus the contributions to E_ρ add up to zero so that E has only z-component.

$$\text{So, } E = \int dE_z = \frac{\rho_s}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\infty}\frac{h\,\rho\,d\phi\,d\rho}{[\rho^2+h^2]^{3/2}}\,a_z$$

$$= \frac{\rho_s h}{4\pi\varepsilon_0}\,2\pi\int_0^{\infty}[\rho^2+h^2]^{-3/2}\cdot\frac{1}{2}\,d(\rho^2)\,a_z$$

actually here, we assume,

$$\rho^2+h^2 = x$$

$$2\rho\,d\rho = dx \quad \text{so, } \rho\,d\rho = \frac{dx}{2} = \frac{1}{2}d(\rho^2)$$

$$= \frac{\rho_s h\cdot \cancel{2\pi}}{2\cdot\cancel{4\pi}\varepsilon_0}\left\{-[\rho^2+h^2]^{-1/2}\right\}_{\rho=0}^{\infty} a_z$$

$$= \frac{\rho_s h}{2\varepsilon_0}\left\{-\cancel{\frac{1}{(\infty^2+h^2)^{1/2}}}^{\,\to 0} + \frac{1}{(0^2+h^2)^{1/2}}\right\} a_z$$

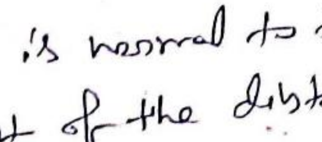
$$\left\{E = \frac{\rho_s}{2\varepsilon_0}a_z\right\} \quad \text{——— (22)}$$

In general, for an infinite sheet of charge

$$\left\{E = \frac{\rho_s}{2\varepsilon_0}a_n\right\} \quad \text{——— (23)}$$

a_n = unit vector normal to the sheet.

→ So by seeing eqn (23), The Electric field is normal to the sheet and it is surprisingly independent of the distance between the sheet and the point of observation P.



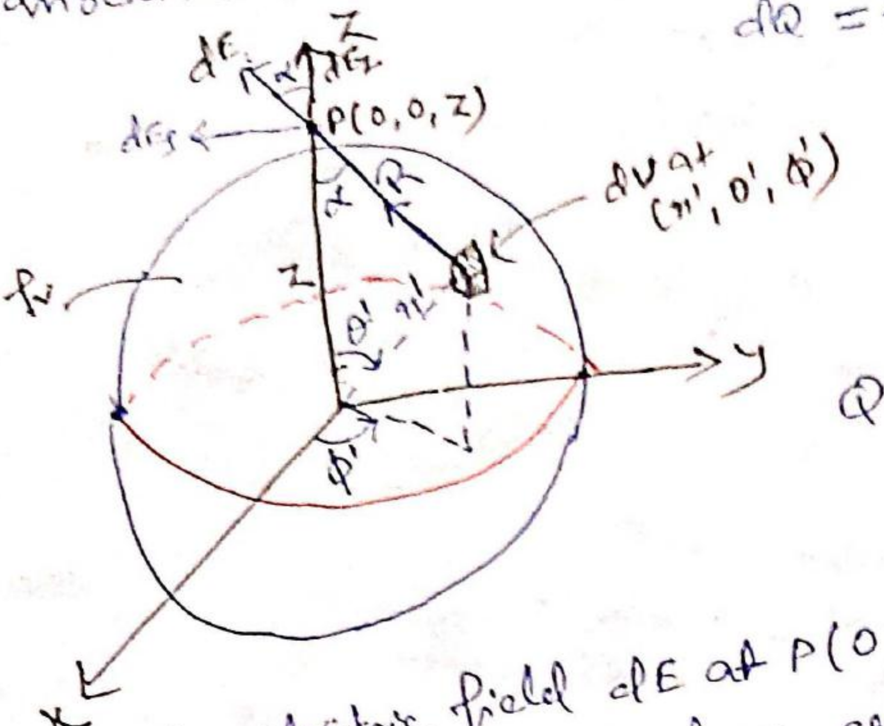
→ So In a parallel plate capacitor, the electric field existing betn the two plates having equal and opposite charge is given by,

$$\left\{E = \frac{\rho_s}{2\varepsilon_0}a_n + \frac{-\rho_s}{2\varepsilon_0}(-a_n) = \frac{\rho_s}{\varepsilon_0}a_n\right\} \text{— (24)}$$

→ Volume charge

Let the volume charge density ρ_v. The charge dQ associated with the elemental volume dv is

$$dQ = \rho_v \, dv$$



Hence the total charge in a sphere of radius 'a' is.

$$Q = \int \rho_v \, dv = \rho_v \int dv$$

$$= \rho_v \frac{4\pi a^3}{3} \quad \text{—— (25)}$$

The electric field dE at $P(0, 0, Z)$ due to elementary volume charge from eqn (8)

$$dE = \frac{\rho_v \, dv}{4\pi\varepsilon_0 R^2} a_R$$

where, $a_R = \cos\alpha \, a_z + \sin\alpha \, a_\rho$

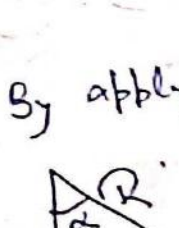
Due to symmetry of the charge distribution, the contribution to E_x or E_y add (or E_ρ) up to zero. So, we are left with only E_z, given by

$$E_z = E \cdot a_z = \int dE \cos\alpha = \frac{\rho_v}{4\pi\varepsilon_0} \int \frac{dv \cos\alpha}{R^2} \quad \text{—— (26)}$$

Now, need to derive expressions for dv, R^2 and $\cos\alpha$

$$dv = r'^2 \sin\theta' \, dr' \, d\theta' \, d\phi' \quad \text{—— (27)}$$

By applying cosine rule, in the triangle

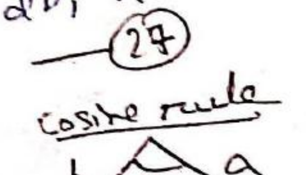


$$R^2 = z^2 + r'^2 - 2zr'\cos\theta'$$

$$r'^2 = z^2 + R^2 - 2zR\cos\alpha$$

—— (28)

Cosine rule



$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

from 28,
$$\begin{cases} \cos\alpha = \dfrac{z^2 + R^2 - r'^2}{2zR} \\[2ex] \cos\theta' = \dfrac{z^2 + r'^2 - R^2}{2zr'} \end{cases} \qquad (29)$$

Since, in volume charge, we have to integrate along r', ϕ' and θ'. but along θ' the integration may be difficult so, instead of θ' we use R.

So, convert, θ' in the form of 'R' by keeping z and r' fixed

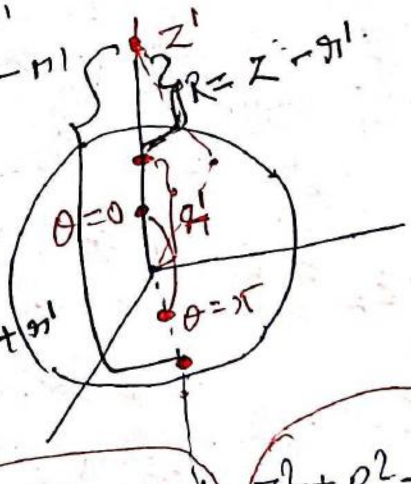
By differentiation, 29(b),

$$\frac{d\cos\theta'}{d\theta'} = \frac{-\cancel{2}R}{\cancel{2}zr'} \cdot \frac{dR}{d\theta'}$$

$$\Rightarrow -\sin\theta' = -\frac{R}{zr'} \cdot \frac{dR}{d\theta'}$$

So, $$\sin\theta' \cdot d\theta' = \frac{R\,dR}{zr'} \qquad (30)$$

and, $$\int_{\theta=0}^{\pi} d\theta' \cdot \sin\theta' = \int_{R=z-r'}^{z+r'} \frac{R}{zr'}\,dR$$



since, $\theta = 0$, $R = z - r'$
$\theta = \pi$, $R = z + r'$

so after substituting (27) to (30) in to (26)

$$E_z = \frac{\rho_v}{4\pi\varepsilon_0} \int_{\phi=0}^{2\pi} \int_{r'=0}^{a} \int_{R=z-r'}^{z+r'} \underbrace{d\phi' \cdot r'^{\cancel{2}} \frac{R\,dR}{zr'}\,dr'}_{dv} \cdot \underbrace{\frac{z^2 + R^2 - r'^2}{2z\cancel{R}}}_{\cos\alpha} \frac{1}{R^2}$$

$$E_z = \frac{\rho_v}{4\pi\varepsilon_0}\cdot 2\pi \int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} r'\, dR\, dr' \cdot \frac{z^2+R^2-r'^2}{2z^2R^2}$$

$$= \frac{\rho_v\cdot 2\pi}{8\pi\varepsilon_0 z^2}\int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} r'\left[1+\frac{z^2-r'^2}{R^2}\right]dR\, dr'$$

integration w.r.t. dR

$$= \frac{\rho_v\, 2\pi}{8\pi\varepsilon_0 z^2}\int_{r'=0}^{a} r'\left[R-\frac{(z^2-r'^2)}{R}\right]_{z-r'}^{z+r'} dr'$$

$$= \frac{\rho_v\cdot 2\pi}{8\pi\varepsilon_0 z^2}\int_0^a r'\left\{\frac{(z+r')^2-(z^2-r'^2)}{(z+r')} - \frac{(z-r')^2-(z^2-r'^2)}{z-r'}\right\} dr'$$

$$= \frac{\rho_v\cdot 2\pi}{8\pi\varepsilon_0 z^2}\int_0^a r'\left\{\frac{\cancel{z^2}+r'^2+2zr'-\cancel{z^2}+r'^2}{z+r'} - \frac{\cancel{z^2}+r'^2+2zr'-\cancel{z^2}+r'^2}{z-r'}\right\} dr'$$

$$= \frac{\rho_v\cdot 2\pi}{8\pi\varepsilon_0 z^2}\int_0^a r'\left\{\frac{(2r'^2+2zr')(z-r')-(2r'^2-2zr')(z+r')}{z^2-r'^2}\right\} dr'$$

$$= \frac{\rho_v\cdot 2\pi}{8\pi\varepsilon_0 z^2}\int_0^a r'\left(\frac{4z^2r'-4r'^3}{z^2-r'^2}\right) dr'$$

$$= \frac{\rho_v\cdot 2\pi}{8\pi\varepsilon_0 z^2}\int_0^a 4r'^2\, dr' = \frac{\rho_v\cdot \cancel{2\pi}}{4\,\cancel{8\pi}\,\varepsilon_0 z^2}\cdot\frac{4}{3}a^3$$

$$= \frac{1}{4\varepsilon_0 z^2}\cdot\frac{4}{3}\rho_v a^3\cdot\frac{\pi}{\pi} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{z^2}\cdot\underbrace{\frac{4}{3}\pi\rho_v a^3}_{Q,\ \text{from eqn } (25)}$$

$$S_3\ \left\{E_z = \frac{Q}{4\pi\varepsilon_0 z^2}\, a_z\right\} \quad \text{———} (31)$$

This result is obtained for E at P(0, 0, z). Owing to the symmetry of the charge distribution, the electric field at P(r, θ, φ) is readily obtained from eqn (31)

$$\boxed{E = \frac{Q}{4\pi\varepsilon_0 r^2}\, a_r} \quad \text{—— (32)}$$

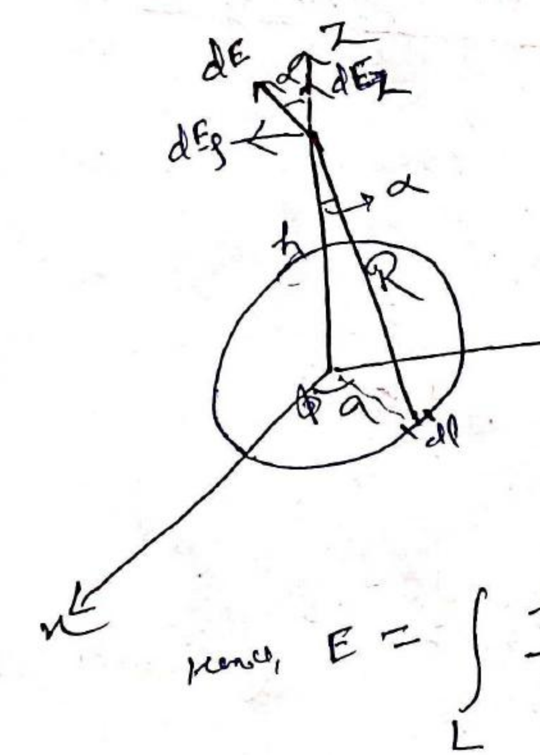
which is identical to the electric field at the same point due to a point charge Q located at the origin on the center of the spherical charge distribution.

Q. A circular ring of radius a carries a uniform charge ρ_L C/m and is placed on the xy-plane with axis the same as the z-axis.

(a) show that $E(0, 0, h) = \dfrac{\rho_L a h}{2\varepsilon_0 [h^2 + a^2]^{3/2}}\, a_z$

(b) What values of h gives the maximum value of E?

(c) If the total charge on the ring is Q, find E as $a \to 0$.



Here, $dl = a\,d\phi$

$R = a(-a_\rho) + h a_z$

$R = |R| = [a^2 + h^2]^{1/2}$

$a_R = \dfrac{R}{R} = \dfrac{R}{|R|}$

or $\dfrac{a_R}{R^2} = \dfrac{R}{|R|^3} = \dfrac{a(-a_\rho) + h a_z}{[a^2 + h^2]^{3/2}}$

Hence, $E = \displaystyle\int_L \frac{\rho_L\, dl}{4\pi\varepsilon_0 R^2}\, a_R$

replace, $dl = a\,d\phi$, integration of φ from 0 to 2π

$$= \frac{\rho_L}{4\pi\varepsilon_0} \int_{\phi=0}^{2\pi} \frac{(-a a_\rho + h a_z)}{[a^2 + h^2]^{3/2}}\, a\,d\phi$$

by symmetry the contribution along a_ρ add up to zero.
Thus we left only z components,

$$E = E_z = \frac{\rho_L}{4\pi\varepsilon_0} \cdot \frac{a h a_z}{[h^2+a^2]^{3/2}} \int_0^{2\pi} d\phi$$

$$= \frac{\rho_L a h \, a_z}{2\varepsilon_0 [h^2+a^2]^{3/2}}$$

(b) $$\frac{d|E|}{dh} = \frac{\rho_L a}{2\varepsilon_0} \left\{ \frac{[h^2+a^2]^{3/2} \cdot \frac{dh}{dh} - \frac{d}{dh}\{[h^2+a^2]^{3/2}\} \cdot h}{[h^2+a^2]^3} \right\}$$

$$= \frac{\rho_L a}{2\varepsilon_0} \left\{ \frac{[h^2+a^2]^{3/2} - \frac{3}{2}(h)\cdot(2h)\,[h^2+a^2]^{1/2}}{[h^2+a^2]^3} \right.$$

for max^m $\frac{d|E|}{dh} = 0$

which means, $[h^2+a^2]^{1/2}\,[h^2+a^2-3h^2] = 0$

$$a^2 - 2h^2 = 0 \Rightarrow h = \pm\frac{a}{\sqrt{2}}$$

(c) Since the charge is uniformaly distributed, the line charge density is $\rho_L = \frac{Q}{2\pi a}$

so $$E = \frac{Qh}{4\pi\varepsilon_0 [h^2+a^2]^{3/2}} a_z$$

as $a \to 0$ $$E = \frac{Q}{4\pi\varepsilon_0 h^2} a_z$$

or in general $$E = \frac{Q}{4\pi\varepsilon_0 r^2} a_R$$

which is same as that of a point charge as one would expect.

⇒ ELECTRIC FLUX DENSITY.

Actually, the electric field intensity is dependent on the medium in which the charge is placed.

So, new vector field D is defined by

$$\boxed{D = \varepsilon_0 E} \quad \text{—— (33)}$$

it is independent of medium

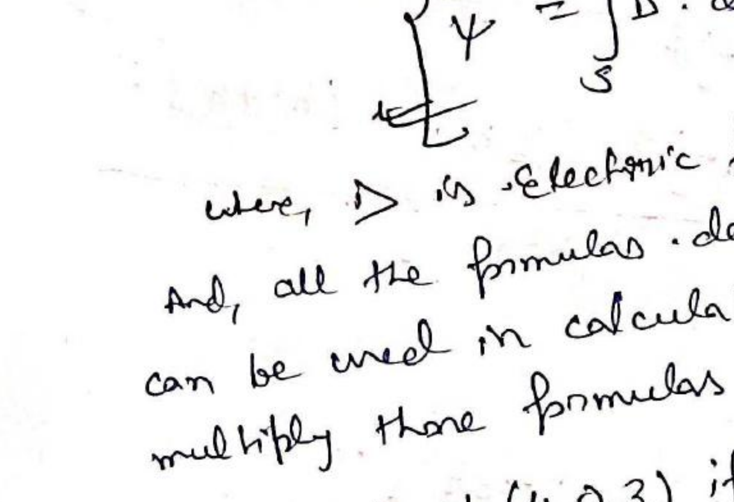
The electric flux Ψ in terms of D, namely

$$\Psi = \int_S D \cdot ds \quad \text{—— (34)}$$

where, D is electronic flux density

And, all the formulas derived for E from Coulomb's law can be used in calculating D, except that we have to multiply those formulas by ε_0.

Q. Determine D at $(4,0,3)$ if there is a point charge -5π mC at $(4,0,0)$ and a line charge 3π mC/m along the y-axis



Let, $D = D_Q + D_L$

D_Q: due to point charge; D_L: due to line charge

$$D_Q = \varepsilon_0 E = \frac{Q}{4\pi R^2} a_r = \frac{Q(r - r')}{4\pi |r - r'|^3}$$

where $r - r' = (4,0,3) - (4,0,0) = (0,0,3)$

$$\text{So } D_Q = \frac{-5\pi \cdot 10^{-3}(0,0,3)}{4\pi |(0,0,3)|^3} = -0.138\, a_z \text{ mC/m}^2$$

Now, $D_L = \dfrac{\rho_L}{2\pi\rho} a_\rho$ Here $a_\rho = \dfrac{(4,0,3) - (0,0,0)}{|(4,0,3) - (0,0,0)|} = \dfrac{(4,0,3)}{5}$

$\rho = |(4,0,3) - (0,0,0)| = 5$ it is min distance from y axis.

$$\text{So, } D_L = \frac{3\pi}{2\pi\, 25}(4a_x + 3a_z) = 0.24 a_x + 0.18 a_z \text{ mC/m}^2$$

$$\text{So } D = D_Q + D_L = 240 a_x + 42 a_z \ \mu\text{C/m}^2$$

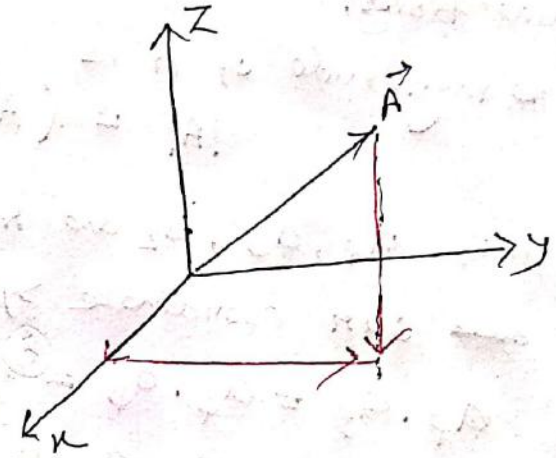
COORDINATE SYSTEMS AND TRANSFORMATION

⇒ CARTESIAN COORDINATES (x, y, z)

Cartesian Coordinates, we all know.

like, a point P can be represented as (x, y, z) where, the range of the coordinate variable x, y, and z are

$$-\infty < x < \infty$$
$$-\infty < y < \infty$$
$$-\infty < z < \infty$$



So, vector $\vec{A}$ can be written in Cartesian (or rectangular) as,

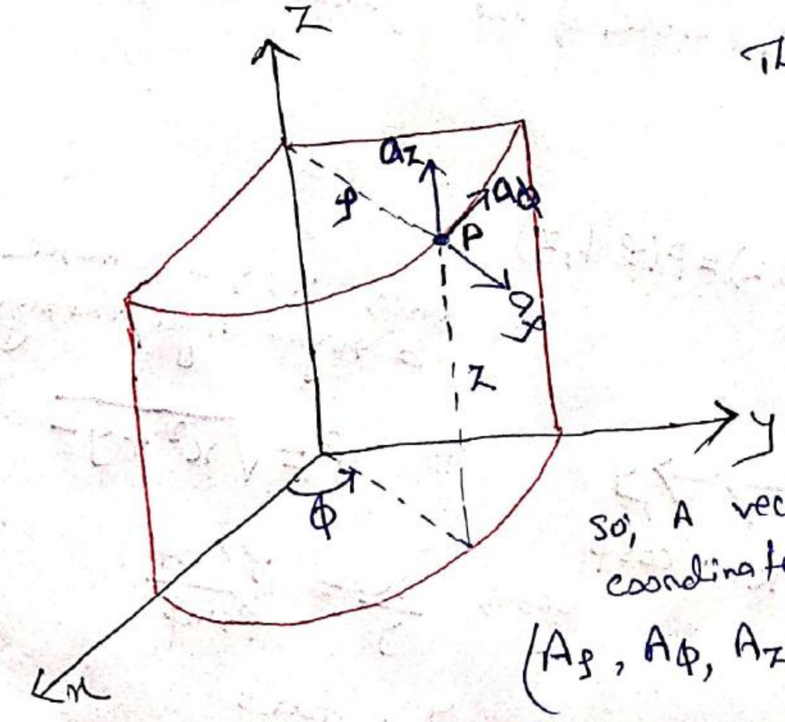
$$\vec{A} = (A_x, A_y, A_z)$$

or,

$$A_x a_x + A_y a_y + A_z a_z$$

⇒ CIRCULAR CYLINDRICAL COORDINATES (ρ, ϕ, z)

A point P in cylindrical coordinates is represented as (ρ, ϕ, z)



The range of the variables are

$$0 \leq \rho < \infty$$
$$0 \leq \phi < 2\pi$$
$$-\infty < z < \infty$$

So, A vector A in cylindrical coordinates can be written as

(A_ρ, A_ϕ, A_z) or, $A_\rho a_\rho + A_\phi a_\phi + A_z a_z$

Here, $\rho \rightarrow$ radius of the cylinder passing through P or the radial distance from the z-axis.

$\phi \rightarrow$ (Azimuthal angle), is measured from the x-axis in the xy-plane.

↳ a_ρ, a_ϕ and a_z are unit vectors in the ρ, ϕ and z directions.

the magnitude of A is

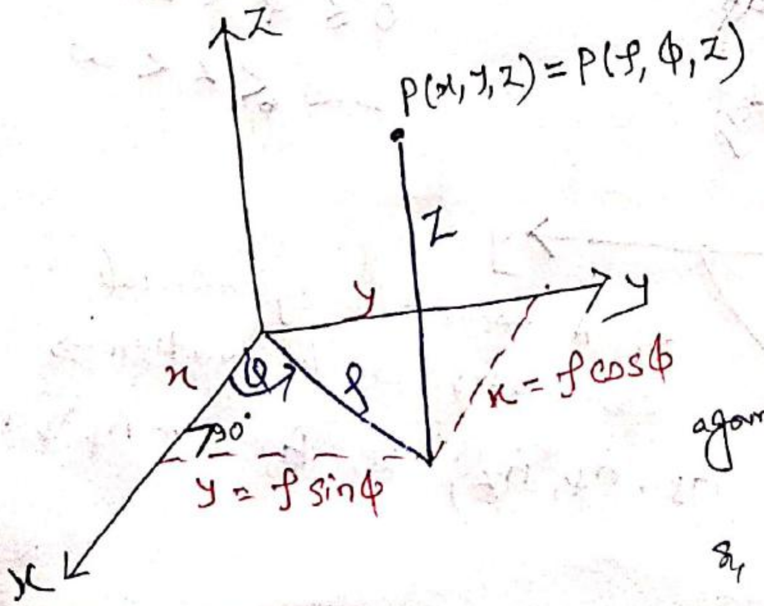
$$|A| = (A_\rho^2 + A_\phi^2 + A_z^2)^{1/2}$$

since, a_ρ, a_ϕ, a_z are mutually perpendicular because our coordinate system are orthogonal

so,
$$a_\rho \cdot a_\rho = a_\phi \cdot a_\phi = a_z \cdot a_z = 1$$
$$a_\rho \cdot a_\phi = a_\phi \cdot a_z = a_z \cdot a_\rho = 0$$

$$a_\rho \times a_\phi = a_z$$
$$a_\phi \times a_z = a_\rho$$
$$a_z \times a_\rho = a_\phi$$

↳ The relationship betn the variables (x, y, z) of the cartesian coordinate system and those of the cylindrical system (ρ, ϕ, z)



since, x, y & ρ make a right angle triangle

so, $\rho = \sqrt{x^2 + y^2}$

again, $\frac{x}{y} = \frac{\rho \cos\phi}{\rho \sin\phi}$.

so, $\frac{y}{x} = \tan\phi$

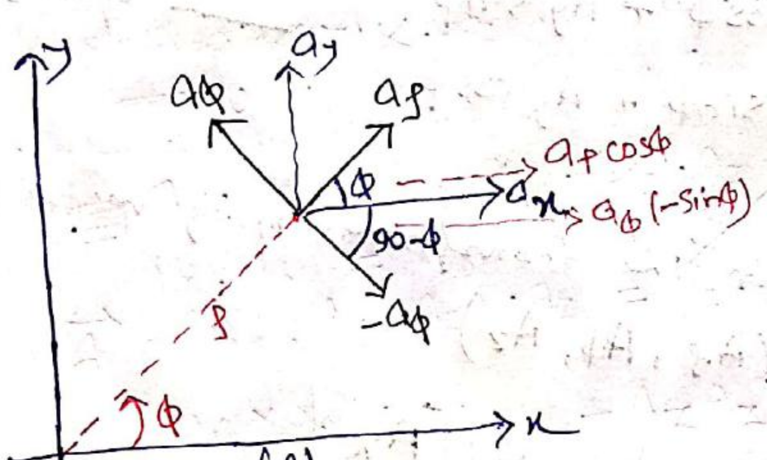
so, $\phi = \tan^{-1}\frac{y}{x}$

so, $\rho = \sqrt{x^2+y^2}$, $\phi = \tan^{-1}\frac{y}{x}$, $z = z$ → transforming (x, y, z) to (ρ, ϕ, z)

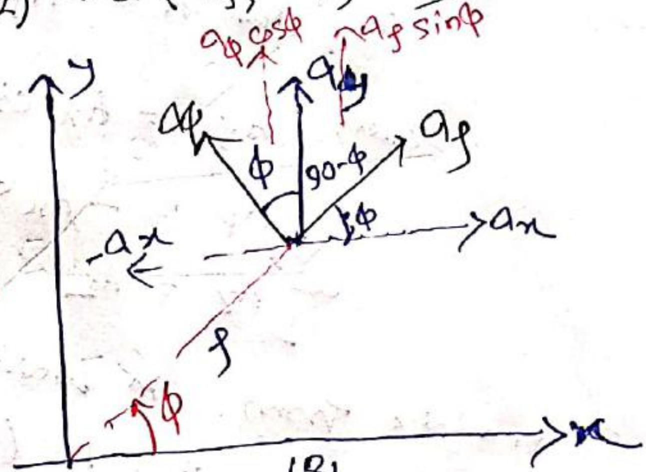
$x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$

transforming a point from $(\rho, \phi, z) \longrightarrow (x, y, z)$

→ The relationships bet^n (a_x, a_y, a_z) and (a_ρ, a_ϕ, a_z) unit vectors



(A)



(B)

The component of a_ρ & a_ϕ along a_x is:

from (B)
$$\begin{cases} a_x = \cos\phi\, a_\rho - \sin\phi\, a_\phi \\ a_y = \sin\phi\, a_\rho + \cos\phi\, a_\phi \\ a_z = a_z \end{cases}$$

, The component of a_ρ & a_ϕ along a_y

, since along z same coordinates

or,
$$\begin{cases} a_\rho = \cos\phi\, a_x + \sin\phi\, a_y \\ a_\phi = -\sin\phi\, a_x + \cos\phi\, a_y \\ a_z = a_z \end{cases}$$

→ from fig (A)

→ from fig (B)

→ The relationship betn (A_x, A_y, A_z) and (A_ρ, A_ϕ, A_z)

like, $A = A_x a_x + A_y a_y + A_z a_z$ in cartesian coordinate

$$= A_x(\cos\phi\, a_\rho - \sin\phi\, a_\phi) + A_y(\sin\phi\, a_\rho + \cos\phi\, a_\phi) + A_z a_z$$

$$A = (A_x \cos\phi + A_y \sin\phi) a_\rho + (-A_x \sin\phi + A_y \cos\phi) a_\phi + A_z a_z$$

or, $A = A_\rho a_\rho + A_\phi a_\phi + A_z a_z$

where,

$$\begin{cases} A_\rho = A_x \cos\phi + A_y \sin\phi \\ A_\phi = -A_x \sin\phi + A_y \cos\phi \\ A_z = A_z \end{cases} \qquad \text{--- eqn (I)}$$

In matrix form, we write the transformation of vector A from (A_x, A_y, A_z) to (A_ρ, A_ϕ, A_z)

from - I

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

the inverse of the transformation $(A_\rho, A_\phi, A_z) \rightarrow (A_x, A_y, A_z)$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

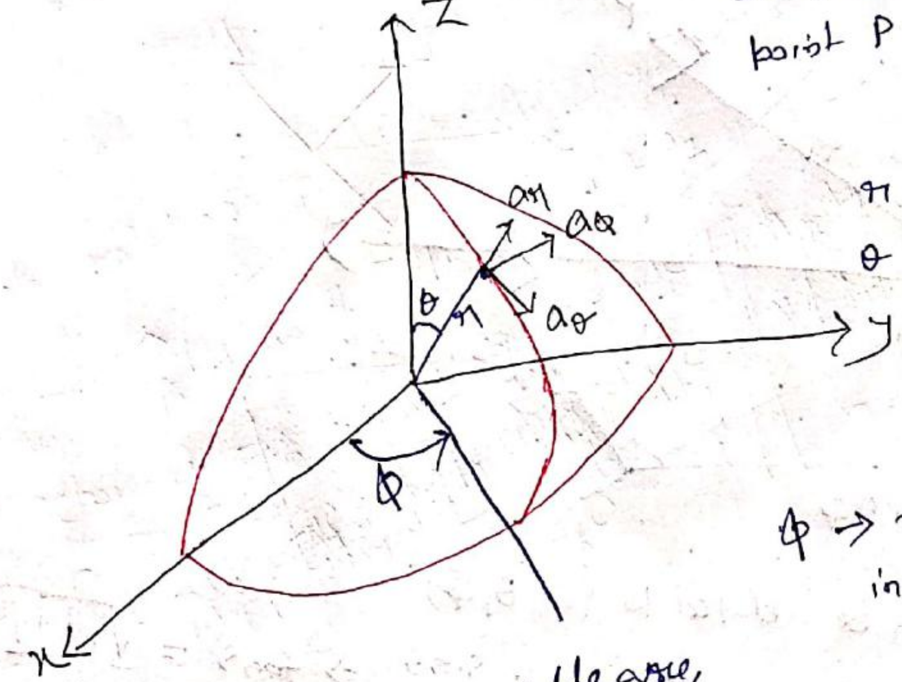
this can be obtain, also from $a_\rho = \cos\phi\, a_x + \sin\phi\, a_y$

$a_\phi = -\sin\phi\, a_x + \cos\phi\, a_y$

$a_z = a_z$

⇒ SPHERICAL COORDINATES (r, θ, ϕ).

In spherical coordinate the point P represent as (r, θ, ϕ)



$r \rightarrow$ radius of sphere

$\theta \rightarrow$ called the colatitude is the angle betⁿ the z-axis and the position vector P

$\phi \rightarrow$ the same azimutal angle in cylindrical coordinates.

So, the range of the variable are,

$0 \leq r < \infty$

$0 \leq \theta \leq \pi$

$0 \leq \phi < 2\pi$

A vector $\vec{A}$ in spherical coordinates may be written as (A_r, A_θ, A_ϕ) or, $A_r a_r + A_\theta a_\theta + A_\phi a_\phi$

and, $|A| = (A_r^2 + A_\theta^2 + A_\phi^2)^{1/2}$

→ The unit vectors a_r, a_θ and a_ϕ are mutually orthogonal

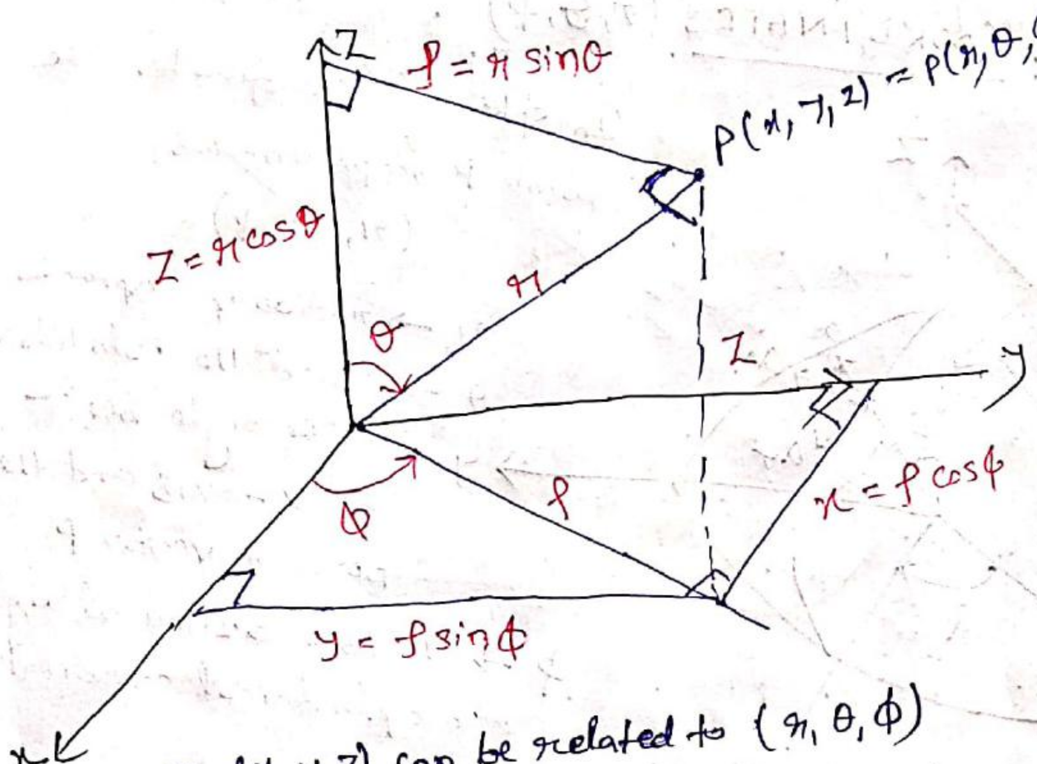
so, $$a_r \cdot a_r = a_\phi \cdot a_\phi = a_\theta \cdot a_\theta = 1$$

$$a_r \cdot a_\theta = a_\theta \cdot a_\phi = a_\phi \cdot a_r = 0$$

$$a_r \times a_\theta = a_\phi$$

$$a_\theta \times a_\phi = a_r$$

$$a_\phi \times a_r = a_\theta$$



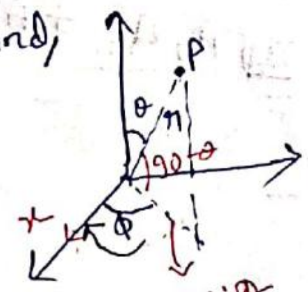
→ The (x, y, z) can be related to (r, θ, φ)

$$r = \sqrt{x^2+y^2+z^2}\ ,\qquad \frac{\rho}{z} = \frac{r\sin\theta}{r\cos\theta} \Rightarrow \tan\theta = \frac{\sqrt{x^2+y^2}}{z}$$

so, $$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$$

and, $$\phi = \tan^{-1}\frac{y}{x}$$

and,



$$x = \rho\cos\phi = r\sin\theta\cos\phi$$

$$y = \rho\sin\phi = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

the component along ~~sinθ~~ x is $r\sin\theta \cdot \cos\phi$

⇒ GAUSS'S LAW – MAXWELL'S EQUATION.

GAUSS'S law states that the total electric flux Ψ through any closed surface is equal to the total charge enclosed by that surface.

then, $$\Psi = Q_{enclosed} \quad \text{—— (35)}$$

ie since $$\Psi = \oint_S d\Psi = \oint_S D \cdot ds \quad \text{—— 36(a)}$$

from the concept of electric flux Density.

and, $$Q_{enclosed} = \int_v \rho_v \cdot dv \quad \text{—— 36(b)}$$

ρ_v = volume charge density

so from 36(a) & 36(b)

$$Q = \oint_S D \cdot ds = \int_v \rho_v dv \quad \text{—— (37)}$$

Note ⇒ Actually, $\Psi = Q_{enclosed}$ is obvious, as we see, $\Psi = \int_S D \cdot ds$ ⇒ If we see the unit of $D = \frac{Q}{4\pi R^2}$ = columb/m^2

so $D \cdot ds$ ⇒ $\frac{C}{m^2} \cdot m^2 = C$

so unit of Ψ is columb ie charge

By Applying divergence theorem

$$\oint_S D \cdot ds = \int_v \nabla \cdot D \, dv \quad \text{—— (38)}$$

so, by comparing (37) & (38)

$$\oint_v \nabla \cdot D \, dv = \int_v \rho_v \, dv$$

so, $$\rho_v = \nabla \cdot D \quad \text{—— (39)}$$

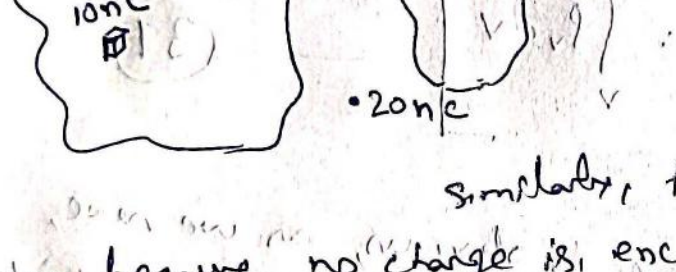
eqn (39) is first of the four Maxwell's equations. It states that the volume charge density is the same as the divergence of the electric flux density.

So, eqn (37) & (39) are basically stating Gauss's law in integral form and differential form. It is alternative statement of coulomb's law.

→ The advantage of Gauss's law is that, it is useful to find the E or D where the charge distribution is symmetrical or not.

like v_1 & v_2 are closed surface (or volumes).

The total flux leaving v_1 is $10-5 = 5\,nC$ because only 10nC and -5nC charges are enclosed by v_1. Although charges 20nC and 15nC outside v_1 do contribute to the flux crossing v_1.

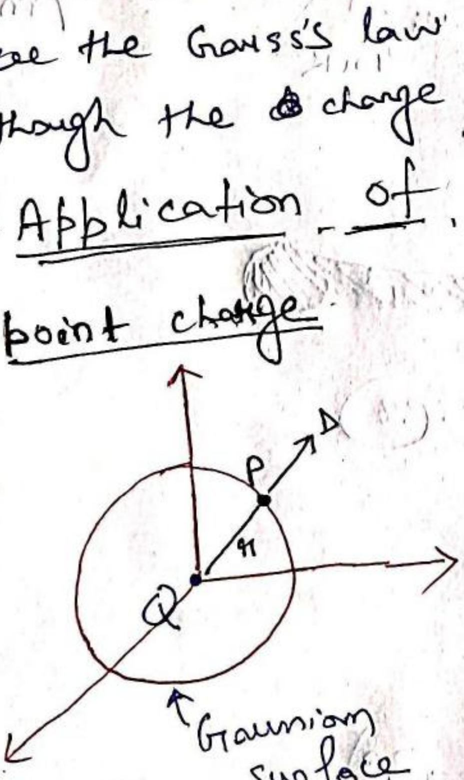


Similarly, the total flux leaving v_2 is zero because no charge is enclosed by v_2. Thus we see the Gauss's law, $\Psi = Q_{enclosed}$ is still obeyed even though the charge distribution is not symmetric.

→ Application of Gauss's Law

point charge

To determine D at a point P, it is easy to see choosing a spherical surface containing P will satisfy symmetry conditions.



Since D is everywhere normal to the Gaussian surface,

i.e, $D = D_r a_r$

Now, applying Gauss's law ($\psi = Q_{enclosed}$)

$$\psi = \oint_S D \cdot ds = Q_{enclosed} = Q$$

$$\Rightarrow D_r \oint_S ds = Q \qquad \text{Here, } \oint_S ds = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi = 4\pi r^2$$

So, $D_r \cdot 4\pi r^2 = Q$

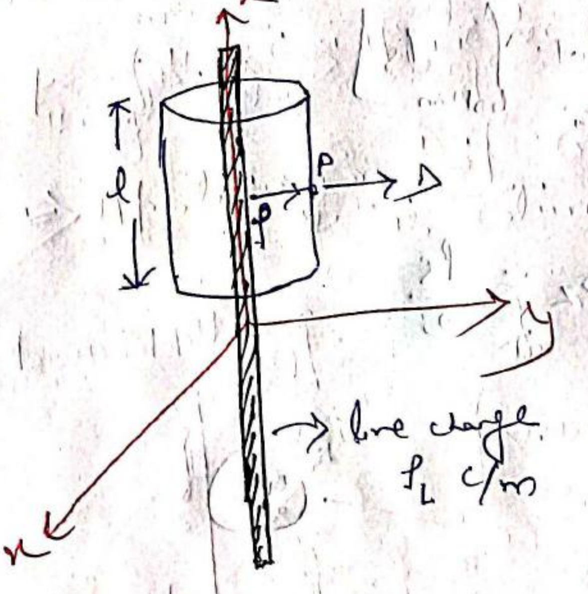
$$\text{i.e, } \left\{ D = D_r = \frac{Q}{4\pi r^2} a_r \right\} \qquad \text{———— (40)}$$

Infinite line charge.

Suppose the infinite line of uniform charge ρ_L C/m lies along the Z-axis. To determine D at a point P, a cylindrical surface containing P is chosen.

Here always $D = D_\rho a_\rho$

by applying Gauss's law to an arbitrary length l of the line.



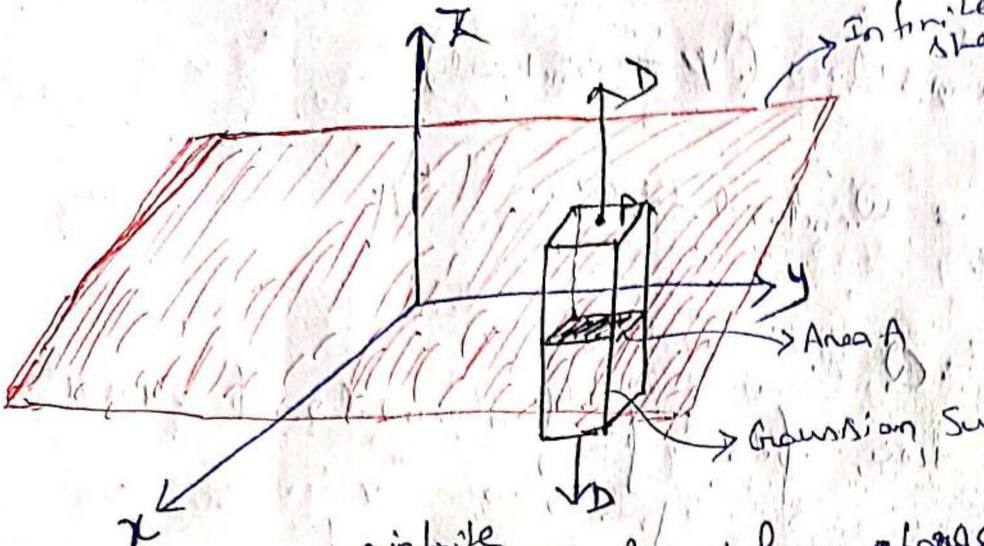
$$Q = \oint D \cdot ds$$

$$\rho_L l = D_\rho \oint_S ds = D_\rho \cdot 2\pi \rho l$$

$$\therefore D_\rho = \frac{\rho_L}{2\pi \rho}$$

$$\text{i.e, } \left\{ D = \frac{\rho_L}{2\pi\rho} a_\rho \right\} \qquad \text{———— (41)}$$

Infinite sheet of charge.



Consider infinite sheet of uniform charge ρ_s C/m² lying on the $z = 0$ plane. To determine D at point P, choose a rectangular box that is cut symmetrically by the sheet of charge.

$$Q = \rho_s \int_S dS = \oint_S D \cdot dS$$

Since D is normal to the sheet, so, $D = D_z a_z$

Since, D has no component along a_x and a_y so, only along z dirn. If the top and bottom area of the box, each has area A,

then,
$$\oint_S D \cdot dS = D_z \left[\int_{top} dS + \int_{bottom} dS \right]$$

$$= D_z [A + A]$$

So, $Q = \rho_s \cdot A = D_z \cdot 2A \Rightarrow D_z = \dfrac{\rho_s}{2}$

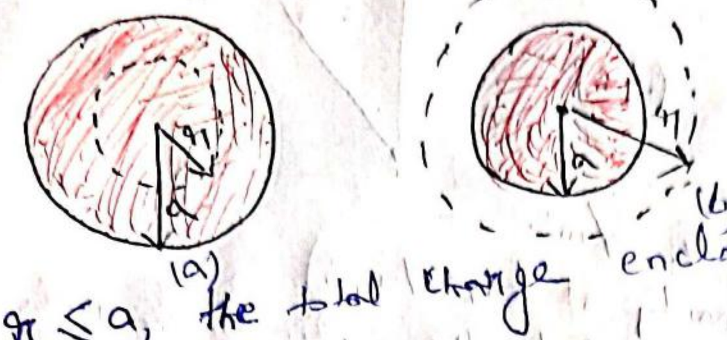
So, $D = \dfrac{\rho_s}{2} a_z$

(iii). and, $E = \dfrac{\rho_s}{2 \varepsilon_0} a_z$

42

Uniformly charged sphere

Consider a sphere of radius a with a uniform charge ρ_v C/m^3.
To determine D everywhere, choose gaussian surface for cases $r \leq a$ and $r \geq a$ separately. i.e. point p once inside the sphere and once outside the sphere.



(a) (b)

for, $r \leq a$, the total charge enclosed by the spherical surface of radius r

$$Q_{enclosed} = \int_v \rho_v \, dv = \rho_v \cdot \frac{4}{3}\pi r^3$$

and, $$\Psi = \oint_s D \cdot ds = D_r \oint_s ds = D_r \cdot 4\pi r^2$$

since, $\Psi = Q_{enclosed}$

$$\Rightarrow D_r \, 4\pi r^2 = \rho_v \cdot \frac{4}{3}\pi r^3$$

$$\left\{ D = \frac{r}{3}\rho_v \, a_r \qquad 0 < r \leq a \right\} \quad (43)$$

for $r \geq a$, the Gaussian surface is (b)

$$Q_{enclosed} = \int_v \rho_v \, dv = \rho_v \frac{4}{3}\pi a^3$$

here, $r > a$ but the charge only up to radius a.

and, $$\Psi = \oint_s D \cdot ds = D_r \, 4\pi r^2$$

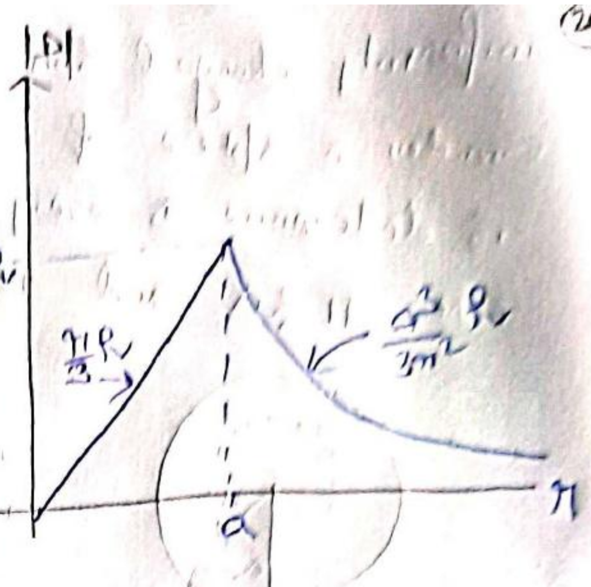
so, when equate, $\Psi = Q_{enclosed}$

$$D_r \, 4\pi r^2 = \rho_v \frac{4}{3}\pi a^3$$

$$\left\{ D = \frac{a^3}{3r^2}\rho_v \, a_r \qquad r \geq a \right\} \quad (44)$$

thus from eqn (43) & (44)

$$D = \begin{cases} \frac{r}{3}\rho_v\, a_r & 0 \le r \le a \\ \frac{a^3}{3r^2}\rho_v\, a_r & r \ge a \end{cases} \qquad (45)$$



Ex Given that $D = z\rho\cos^2\phi\, a_z$ C/m²
calculate the charge density at $(1, \pi/4, 3)$
and the total charge enclosed by the cylinder of radius 1m with $-2 \le z \le 2$ m.

Soln charge density $\rho_v = \nabla\cdot D = \frac{\partial D_z}{\partial z} = \rho\cos^2\phi$ C/m³

so at $(1, \pi/4, 3)$, $\rho_v = 1\cdot\cos^2(\pi/4)$ C/m³
$= 0.5$ C/m³

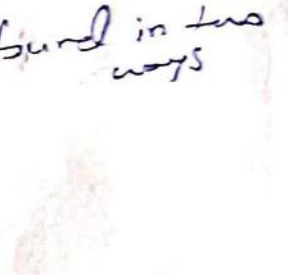
(ii) the total charge enclosed by the cylinder can be found in two ways

↳ directly we can find volume charge.

$$Q = \int \rho_v\, dv = \int \rho\cos^2\phi\, \rho\, d\phi\, d\rho\, dz$$
$$= \int_{z=-2}^{2} dz \int_{\phi=0}^{2\pi} \cos^2\phi\, d\phi \int_{\rho=0}^{1} \rho^2\, d\rho = 4(\pi)(1/3) = \frac{4\pi}{3}\ \text{C}$$

↳ we can use Gauss' law

$$Q = \Psi = \oint D\cdot dS = \left[\int_s + \int_t + \int_b\right] D\cdot dS = \Psi_s + \Psi_t + \Psi_b$$



Since, D does not have component along a_ρ,

so $\Psi_s = 0$.

for Ψ_t, $dS = \rho\, d\phi\, d\rho\, a_z$ so,

$$\Psi_t = \int_{\rho=0}^{1}\int_{\phi=0}^{2\pi} z\rho\cos^2\phi\, \rho\, d\phi\, d\rho \Big|_{\text{at } z=2}$$

$$\Psi_t = 2\int_0^1 \rho^2 d\rho \int_0^{2\pi} \cos^2\phi \, d\phi = 2\left(\frac{1}{3}\right)\pi = \frac{2\pi}{3}$$

and for Ψ_b, $ds = -\rho \, d\phi \, d\rho \, a_z$

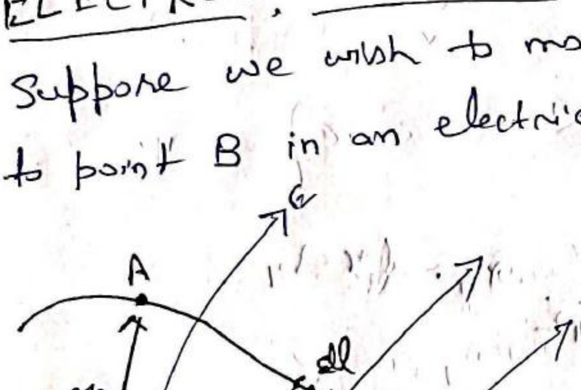
$$\Psi_b = -\int_{\rho=0}^{1}\int_{\phi=0}^{2\pi} z\rho\cos^2\phi \, \rho \, d\phi \, d\rho \Big|_{z=-2}$$

$$= 2\int_0^1 \rho^2 d\rho \int_0^{2\pi} \cos^2\phi \, d\phi = \frac{2\pi}{3}$$

Thus, $Q = \Psi = 0 + \frac{2\pi}{3} + \frac{2\pi}{3} = \frac{4\pi}{3}$ C

⇒ **ELECTRIC POTENTIAL**

Suppose we wish to move a point charge Q from point A to point B in an electric field E as shown in figure.



From Coulomb's Law, the force on Q is $F = QE$

So, the work done in displacing the charge by dl is

$$dW = -F \cdot dl = -QE \cdot dl \qquad (46)$$

the negative sign indicates that the work is being done by an external agent. Thus the total work done, or the potential energy required in moving Q from A to B, is

$$W = -Q\int_A^B E \cdot dl \qquad (47)$$

Dividing W by Q gives the potential energy per unit charge. This quantity, denoted by V_{AB} is known as the potential difference bet^n points A and B.

Then $$\left\{ V_{AB} = \frac{W}{Q} = -\int_A^B E \cdot dl \right\} \text{ Joule/C or volt} \quad \text{--- (48)}$$

→ In V_{AB}, A is initial point while B is the final point.

→ If V_{AB} is negative, there is a loss in potential energy in moving Q from A to B; this implies that the work is being done by the field. However, if V_{AB} is +ve, there is gain in potential energy in the movement; an external agent performs the work.

→ V_{AB} is independent of the path taken.

like, E for point charge
$$E = \frac{Q}{4\pi\epsilon_0 r^2} a_r$$

So, eq^n (48), $$V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2} a_r \cdot dr\, a_r$$

$$\left\{ V_{AB} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right] \right. \quad \text{--- (49)}$$

or, $$\left\{ V_{AB} = V_B - V_A \right. \quad \text{--- (50)}$$

where V_B and V_A are the potentials (or absolute potentials) at B and A respectively. So, V_{AB} regarded as potential at B with reference to A.

if reference, A, to be chosen at infinity, then
$$V_A = \frac{Q}{4\pi\epsilon_0 r_A} \quad \text{since } r_A \to \infty \text{ so } V_A \to 0$$

So, $$\left\{ V = \frac{Q}{4\pi\epsilon_0 r} \right\} \quad \text{--- (51)}$$

S) The potential at any point is the potential difference betn that point and a chosen point (or reference point) at which the potential is zero.

ie, the work done per unit charge by an external agent in transferring a test charge from infinity to that point,

$$V = -\int_{\infty}^{r} E \cdot dl \qquad (52)$$

If the point charge Q not located at the origin but at a point whose position vector is r'.

So,

$$V(r) = \frac{Q}{4\pi\varepsilon_0 |r - r'|} \qquad (53)$$

Similarly, for n point charges $Q_1, Q_2, \ldots, Q_n$ located at points with position vectors $r_1, r_2, \ldots, r_n$, the potential at r is

$$V(r) = \frac{Q_1}{4\pi\varepsilon_0 |r - r_1|} + \frac{Q_2}{4\pi\varepsilon_0 |r - r_2|} + \cdots + \frac{Q_n}{4\pi\varepsilon_0 |r - r_n|}$$

or,

$$V(r) = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{n} \frac{Q_n}{|r - r_n|} \quad \text{for point charge} \qquad (54)$$

for continuous charge distributions, we replace Q_k with charge elements, so,

$$V(r) = \frac{1}{4\pi\varepsilon_0} \int_L \frac{\rho_L(r')\,dl'}{|r - r'|} \quad \text{(line charge)}$$

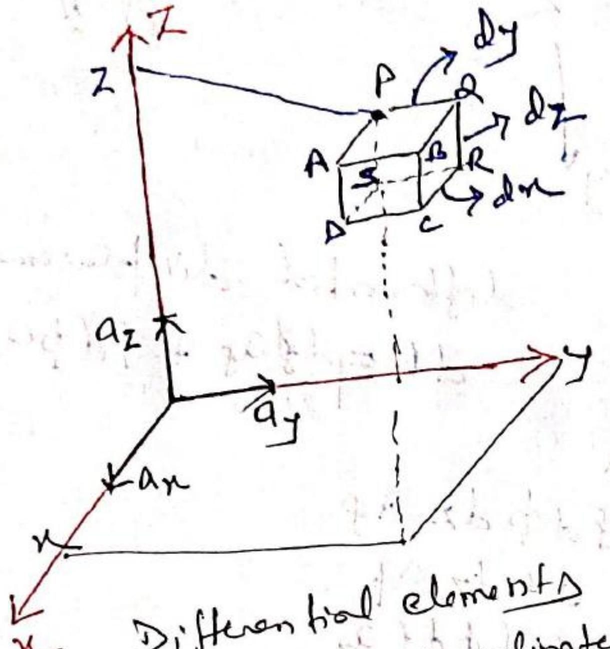
$$V(r) = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\rho_S(r')\,dS'}{|r - r'|} \quad \text{(surface charge)} \qquad (55)$$

$$V(r) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho_v(r')\,dv'}{|r - r'|} \quad \text{(volume charge)}$$

VECTOR CALCULUS

=> DIFFERENTIAL LENGTH, AREA, AND VOLUME.

↳ Cartesian Coordinates



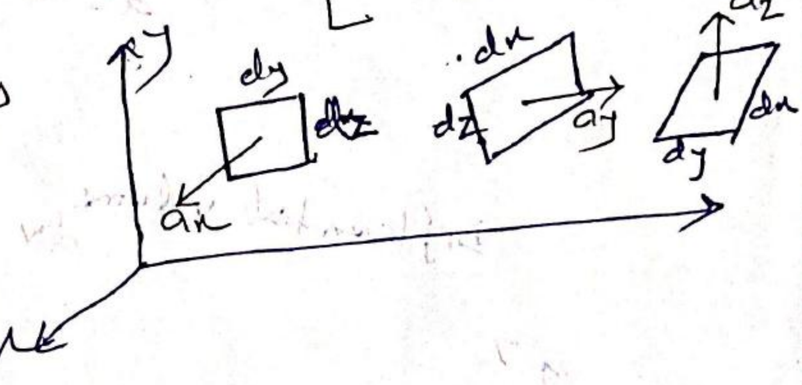
Differential elements in the cartesian coordinates system.

↳ Differential displacement

$$d\mathbf{l} = dx\,a_x + dy\,a_y + dz\,a_z$$

↳ Differential normal area given by

$$d\mathbf{S} = dy\,dz\,a_x$$
$$= dx\,dz\,a_y$$
$$= dx\,dy\,a_z$$



↳ Differential volume is given by

$$dv = dx\,dy\,dz$$

So, the, $d\mathbf{l}$ & $d\mathbf{S}$ are the vector quantity but dv is the scalar quantity.

↳ to move from D to Q would mean that

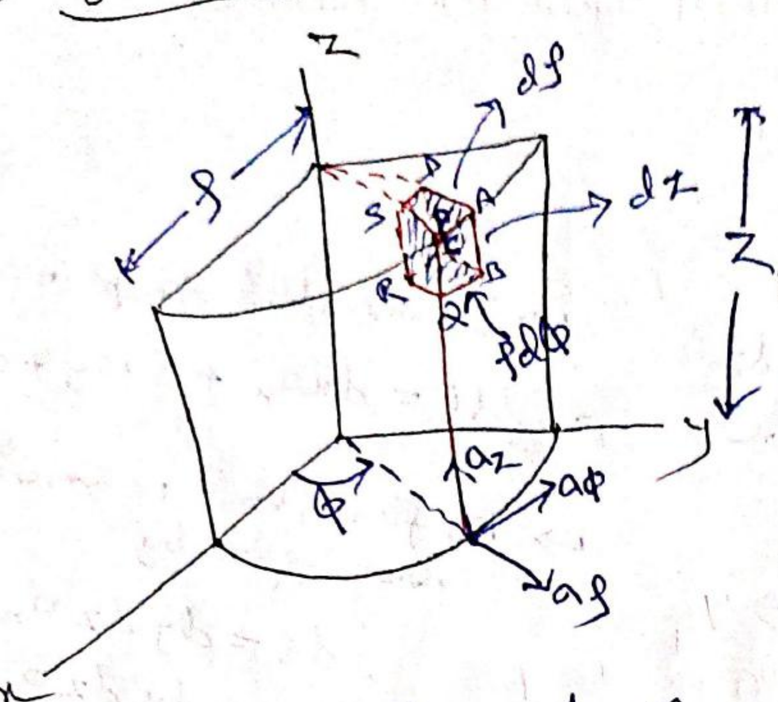
$$d\mathbf{l} = dx\,a_x + dy\,a_y + dz\,a_z$$

↳ The differential surface (or area) element $d\mathbf{S}$ defined as

$$d\mathbf{S} = dS\,a_n$$

where dS is the area of the surface element and a_n is the unit vector normal to the surface dS (and directed away from the volume if dS is part of the surface describing the volume) like, for surface ABCD, $ds = dy\,dz\,a_x$ whereas, for surface PQRS, $ds = -dy\,dz\,a_x$

↳ Cylindrical Coordinates.



$\rho\,d\phi$; $\sin d\phi = \frac{QB}{\rho}$

$QB = \rho\,d\phi$

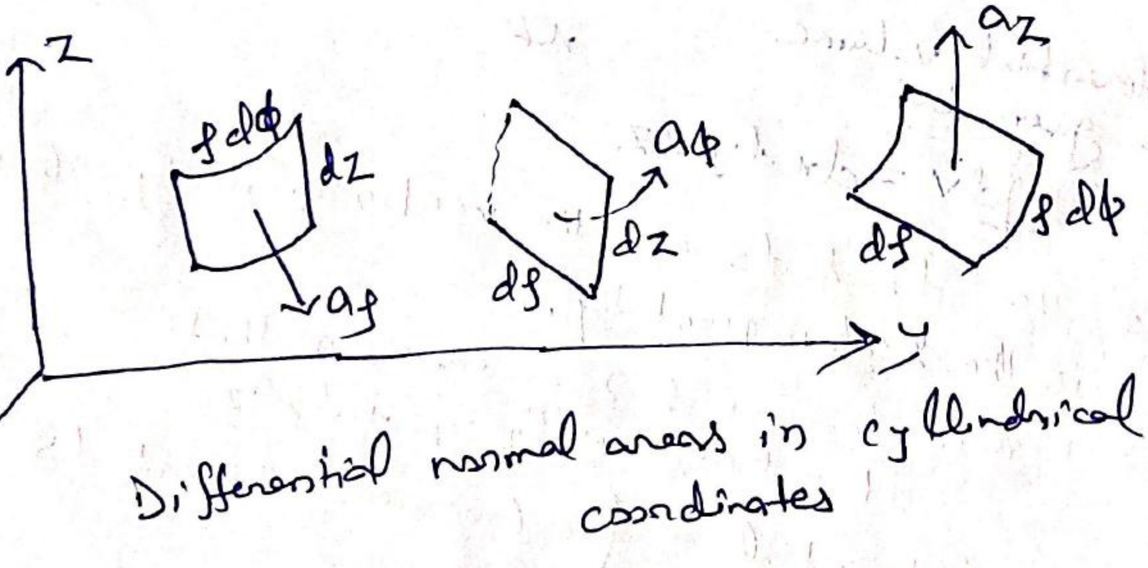
differential displacement

$$dl = d\rho\, a_\rho + \rho\, d\phi\, a_\phi + dz\, a_z$$

Differential normal area

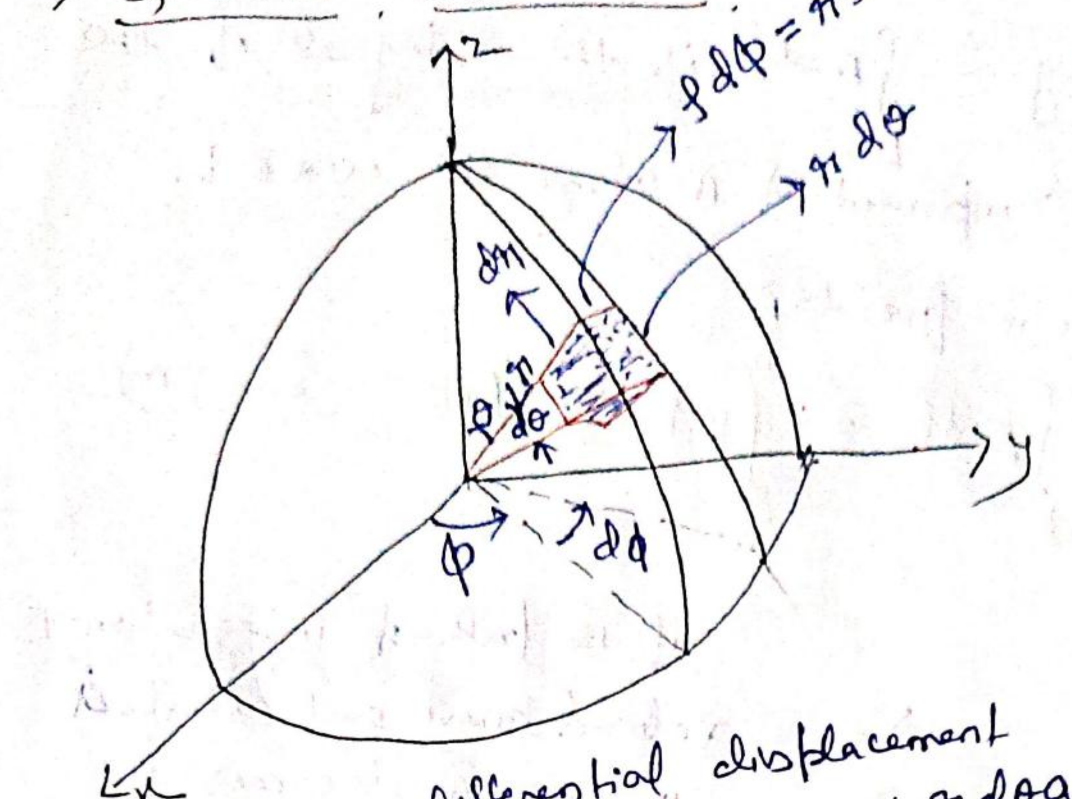
$$ds = \rho\, d\phi\, dz\, a_\rho$$
$$d\rho\, dz\, a_\phi$$
$$\rho\, d\phi\, d\rho\, a_z$$

Differential volume $dv = \rho\, d\rho\, d\phi\, dz$



Differential normal areas in cylindrical coordinates

→ Spherical Coordinates



So, the differential displacement

$$dl = dr\, a_r + r\, d\theta\, a_\theta + r \sin\theta\, d\phi\, a_\phi$$

∴ the differential normal area ds,

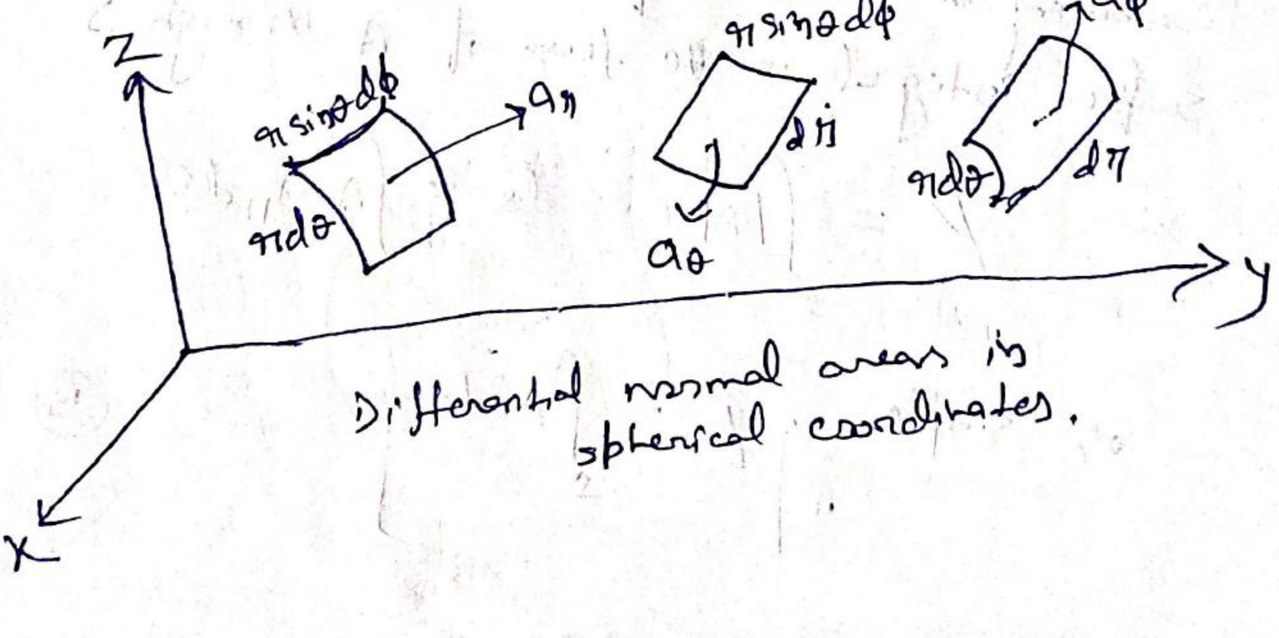
$$ds = r^2 \sin\theta\, d\theta\, d\phi\, a_r = r\, d\theta \cdot r \sin\theta\, d\phi \cdot a_r$$

$$r \sin\theta\, dr\, d\phi\, a_\theta = r \sin\theta\, d\phi \cdot dr\, a_\theta$$

$$r\, dr\, d\theta\, a_\phi$$

The differential volume

$$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$$



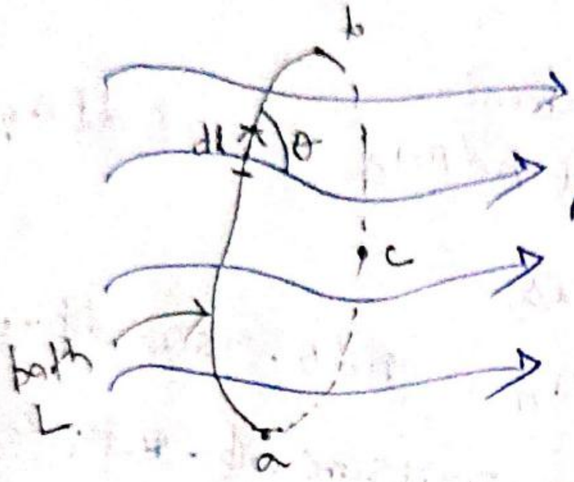
Differential normal areas in spherical coordinates.

⇒ LINE, SURFACE, AND VOLUME INTEGRALS

The line integral $\int_L A \cdot dl$ is the integral of the tangential component of A along the curve L.

The Given vector field A

$$\int_L A \cdot dl = \int_a^b |A| \cos\theta \, dl \qquad \text{— (A)}$$



If the path of integration is a closed curve such as abca then, eqn (A) becomes a closed contour integral

$$\oint_L A \cdot dl$$

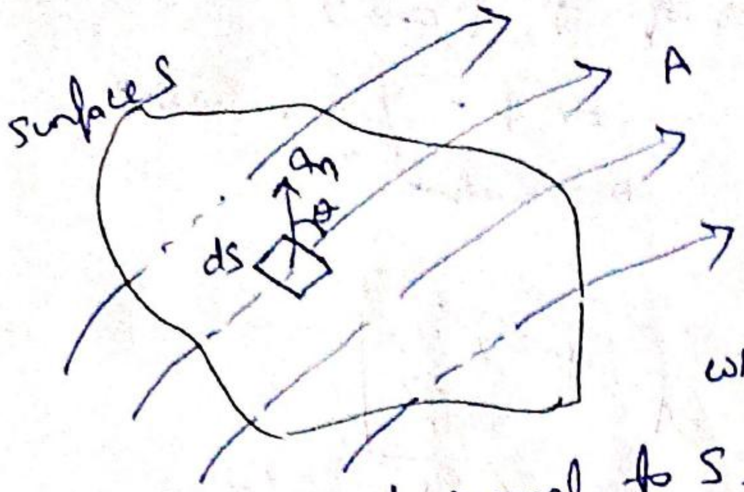
which is called circulation of A around L.

↳ · Given a vector field A, continuous in a region containing the smooth surface S, we define the surface integral or the flux of A through S

$$\Psi = \int_S |A| \cos\theta \, dS = \int_S A \cdot a_n \, dS$$

or, simple

$$\left\{ \Psi = \int_S A \cdot dS \right\} \qquad \text{— (B)}$$



where at any point on S,

a_n is the unit normal to S.

S, for closed surface, eq (B) becomes

$$\left\{ \psi = \oint_S A \cdot dS \right\}$$

it is referred to as the net outward flux of A from S.

→ Important point is, the closed path defines a open surface whereas a closed surface defines a volume

So, we define the integral,

$\int_v \rho_v \cdot dv$ as the volume integral of the scalar ρ_v over a volume v.

⇒ DEL OPERATOR

The del operator ∇, is the vector differential operator.

In cartesian coordinate

$$\left\{ \nabla = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z \right. \quad \text{——— (C)}$$

to obtain, ∇ in terms of ρ, ϕ and z

since, $\rho = \sqrt{x^2+y^2}$, $\tan\phi = \frac{y}{x}$,

$$\nabla = a_\rho \frac{\partial}{\partial \rho} + a_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + a_z \frac{\partial}{\partial z}$$

Just, replace, $dl = \underline{d\rho}\, a_\rho + \underline{\rho\, d\phi}\, a_\phi + \underline{dz}\, a_z$

for Cylindrical Coordinates

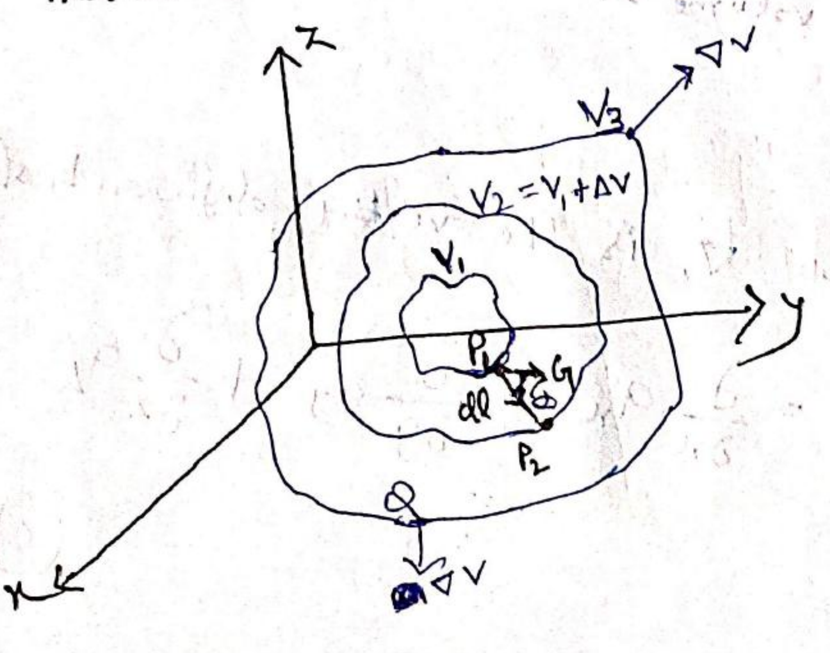
del operator,

$$\nabla = a_r \frac{\partial}{\partial r} + a_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + a_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

differential length in spherical, $dr = \underline{dr}\, a_r + \underline{r\, d\theta}\, a_\theta + \underline{r\sin\theta\, d\phi}\, a_\phi$

⇒ Gradient of a Scalar

The gradient of a scalar field V is a vector that represents both the magnitude and the direction of maximum space rate of increase of V.



The mathematical expression for the gradient can be obtained by evaluating the difference in the field dV betⁿ P_1 and P_2, where V_1, V_2 and V_3 are contours on which V is constant.

from Calculus,

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$= \left(\frac{\partial V}{\partial x}a_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z\right)\cdot\left(dx\,a_x + dy\,a_y + dz\,a_z\right)$$

for convenience,

let, $G = \frac{\partial V}{\partial x}a_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z$

so, $dV = G\cdot dl = G\cos\theta\, dl$

or, $\frac{dV}{dl} = G\cos\theta$

Here, dl is the differential displacement from P_1 to P_2 and θ is the angle betn G and dl.

so, $\left.\frac{dV}{dl}\right|_{max} = \frac{dV}{dn} = G$

at $\theta = 0$; $\frac{dV}{dn}$ → normal derivative

so, G has its magnitude and direction as those of the maximum rate of change of V.

so, By definition, G is the gradient of V.

so, $\boxed{\text{grad } V = \nabla V = \frac{\partial V}{\partial x}a_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z}$

so, grad V in,

Cartesian, $\nabla V = \frac{\partial V}{\partial x}a_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z$

Cylindrical, $\nabla V = \frac{\partial V}{\partial \rho}a_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}a_\phi + \frac{\partial V}{\partial z}a_z$

Spherical, $\nabla V = \frac{\partial V}{\partial r}a_r + \frac{1}{r}\frac{\partial V}{\partial \theta}a_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}a_\phi$

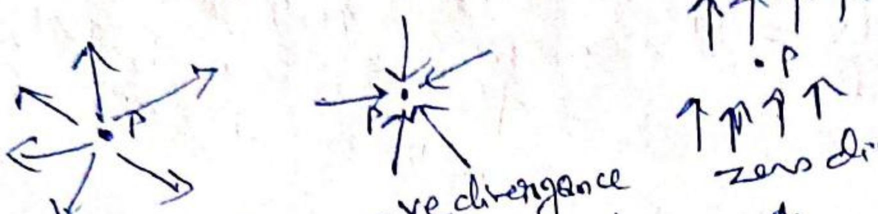
→ Divergence of a vector and Divergence theorem

The divergence of A at a given point P is the outward flux per unit volume as the volume shrinks about P.

where, A (~~[illegible]~~) is vector field.

Hence,
$$\operatorname{div} A = \nabla\cdot A = \lim_{\underbrace{\Delta V \to 0}_{\text{volume tends to point}}} \frac{\oint_S A\cdot ds}{\Delta V} \quad \begin{matrix}\rightarrow \text{net flux}\\ \rightarrow \text{volume}\end{matrix}$$

Here, $\oint_S A\cdot ds$ = net outflow of the flux (from line integral)

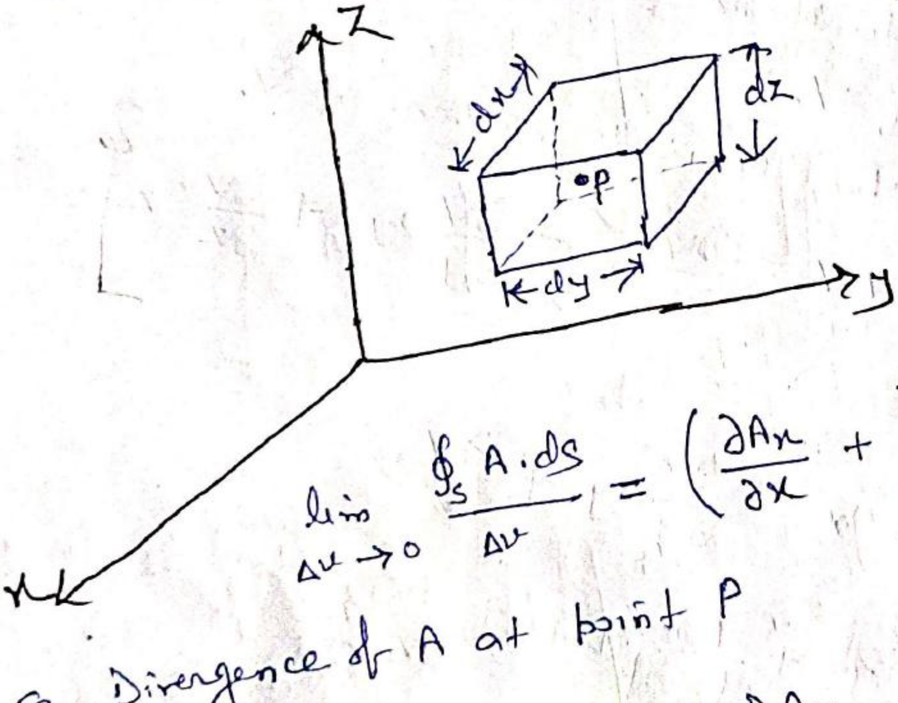


+ve divergence (Source point)

-ve divergence (sink point)

zero divergence.

ΔV is the volume enclosed by closed surface S in which p is located.



$$\lim_{\Delta V \to 0} \frac{\oint_S A\cdot ds}{\Delta V} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\Bigg|_{at\ P} = \nabla\cdot A$$

So, Divergence of A at point P

in cartesian,
$$\nabla\cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

in cylindrical,

$$\nabla \cdot A = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

in spherical,

$$\nabla \cdot A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

The properties of divergence of a vector field.

(i) It produces a scalar field ($\nabla \cdot A \rightarrow$ dot product of two vector field is scalar)

(ii) $\nabla \cdot (A+B) = \nabla \cdot A + \nabla \cdot B$

(iii) $\nabla \cdot (VA) = V\nabla \cdot A + A \cdot \nabla V$

→ since,

$$\oint_S A \cdot ds = \sum_k \oint_{S_k} A \cdot ds = \sum_k \frac{\oint_{S_k} A \cdot ds}{\Delta v_k} \Delta v_k = \int_v \nabla \cdot A \, dv$$

$\nabla \cdot A$

the volume, v divided in to large numbers of small cells. If the kth cell has volume Δv_k and is bounded by surface S_k.

so,

$$\oint_S A \cdot ds = \int_v \nabla \cdot A \, dv$$

This is divergence theorem,

→ The divergence theorem states that the total outward flux of a vector field A through the closed surface S is the same as the volume integral of the divergence of A.

→ Curl of a vector and Stokes's theorem.

The curl of A is an axial (or rotational) vector whose magnitude is the maximum circulation of A per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.

So,

$$\text{Curl } A = \nabla \times A = \left(\lim_{\Delta S \to 0} \frac{\oint_L A \cdot dl}{\Delta S} \right)_{max} a_n$$

where, the area ΔS is bounded by the curve L and a_n is the unit vector normal to the surface ΔS and is determined using the right-hand rule.

for cartesian coordinates

$$\nabla \times A = \begin{vmatrix} a_x & a_y & a_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

or,

$$\nabla \times A = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right] a_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right] a_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right] a_z$$

from, cylindrical,

$$\nabla \times A = \frac{1}{\rho} \begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial \rho} & \partial/\partial \phi & \partial/\partial z \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

for spherical coordinates,

$$\nabla \times A = \frac{1}{r^2 \sin\theta} \begin{vmatrix} a_r & r a_\theta & r\sin\theta\, a_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r\sin\theta\, A_\phi \end{vmatrix}$$

properties of curl,

(i) The curl of a vector field is another vector field

(ii) $\nabla \times (A+B) = \nabla \times A + \nabla \times B$

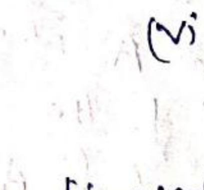
(iii) $\nabla \times (A \times B) = A(\nabla \cdot B) - B(\nabla \cdot A) + (B \cdot \nabla A - (A \cdot \nabla)B$

(iv) $\nabla \times (VA) = V\nabla \times A + \nabla V \times A$

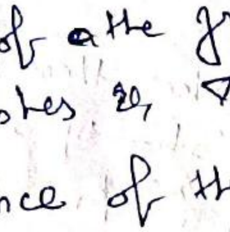
(v) The divergence of a curl of a vector field vanishes ie, $\nabla \cdot (\nabla \times A) = 0$

(vi) The curl of ~~a~~ the gradient of a scalar field vanishes ie, $\nabla \times \nabla V = 0$

The physical significance of the curl of a vector field is that the curl provides the maximum value of the circulation of the field per unit area (or circulation density) and indicates the direction along which this maximum value occurs. The curl of a vector field A at a point P may be regarded as a measure of the circulation or how much the field curls around P.



curl at point P points out of the page



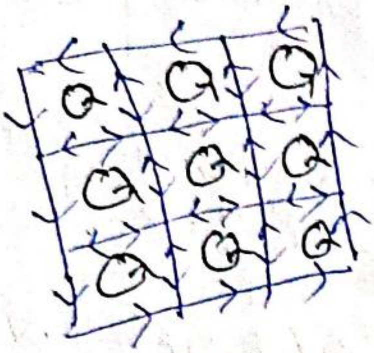
curl at P is zero

→ Stockes's theorem

Since, we can write,

$$\oint_L A\cdot dl = \sum_K \oint_{L_K} A\cdot dl = \sum_K \frac{\oint_{L_K} A\cdot dl}{\Delta S_K}\,\Delta S_K$$

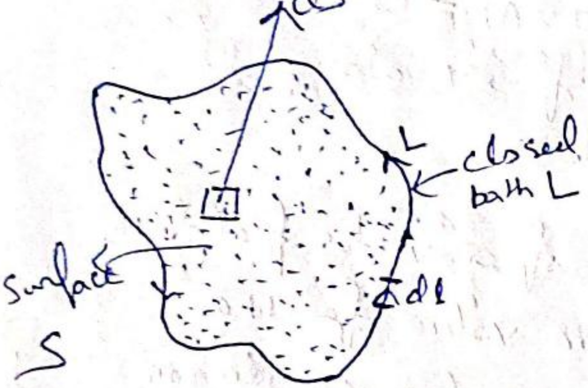
$\nabla \times A$



The surface S is subdivided into a large number of cells as in figure. If the kth cell has surface area ΔS_K and is bounded by path L_K.

$$\boxed{\oint_L A\cdot dl = \int_S (\nabla\times A)\cdot dS}$$

This is called stoke's theorem



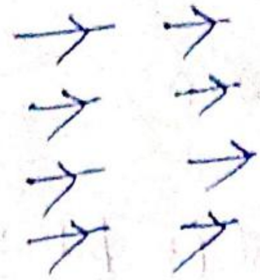
Stockes theorem states that the circulation of a vector field A around a (closed) path L is equal to the surface integral of the curl of A over the open surface S bounded by L provided that A and $\nabla \times A$ are continuous on S.

The direction of dl and dS must be chosen using the right-hand rule or right-handed screw rule.

⇒ Classification of Vector fields.

A vector field is uniquely characterized by its divergence and curl. Means Both required to characterized the field.

Ex (i) $\nabla \cdot A = 0, \quad \nabla \times A = 0$

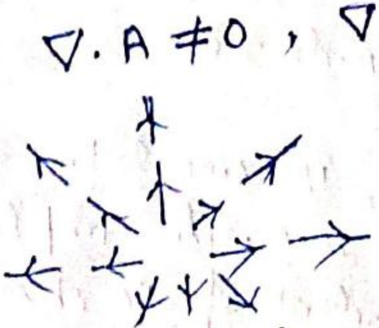


Since, there is not source or sink and no rotation so, its divergence and curl both zero of vector field.

like, $A = k\, a_x$

so, $\nabla \cdot A = 0, \quad \nabla \times A = 0$

(ii) $\nabla \cdot A \neq 0, \quad \nabla \times A = 0$

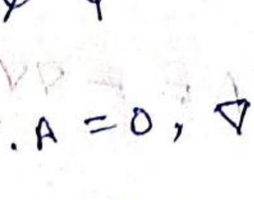


Divergence is not zero. So it is source or sink or, but curl is zero so, no rotation

like, $A = k r$

so, $\nabla \cdot A = 3k, \quad \nabla \times A = 0$

(iii) $\nabla \cdot A = 0, \quad \nabla \times A \neq 0$



Divergence is zero so, no source or sink. but curl is not zero so, rotation vector

like, $A = k \times r$

so, $\nabla \cdot A = 0, \quad \nabla \times A = 2k$

(iv) $\nabla \cdot A \neq 0, \quad \nabla \times A \neq 0$

It has both the property. like vector field behaves as a source or sink as well as rotation.

$A = k \times r + c r, \quad \nabla \times A = 2k$

so, $\nabla \cdot A = 3c$,

$\Rightarrow$ A vector field A is said to be solenoidal (or divergenceless) if $\nabla \cdot A = 0$ i.e., is obvious

So, from divergence theorem,

$$\oint_S A \cdot ds = \int_V \nabla \cdot A \, dv = 0$$

Here, if, $\nabla \cdot A = 0$

then, $\oint_S A \cdot ds = 0$ and $F = \nabla \times A$

=> A vector A is said to be irrotational (or potential)

if $\nabla \times A = 0$

from stokes theorem, $\int_S (\nabla \times A) \cdot ds = \oint_L A \cdot dl = 0$

Thus in a irrotational field A, the circulation of A around a closed path is identically zero. This implies that the line integral of A is independent of the chosen path.

So, if $\nabla \times A = 0$

then, $\oint_L A \cdot dl = 0$ and, $A = -\nabla V$

=> Laplacian of a Scalar

The Laplacian of a Scalar field V, written as $\nabla^2 V$, is the divergence of the gradient of V.

actually Laplacian V $= \nabla \cdot \underbrace{\nabla V}_{\text{gradient of } V} = \nabla^2 V$

divergence of the gradient of V

So, Laplacian is a single operator which is the composite of gradient and divergence operators.

So Laplacian V $= \nabla \cdot \nabla V$

$$= \left[\frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z\right] \cdot \left[\frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z\right]$$

So, $$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$